# Connections between graphs and matrix spaces

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# What are the matrix spaces?

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- Matrix spaces are linear spaces spanned by **matrices**.
- Formally, let $\mathrm{M}(m \times n, \mathbb{F})$ denote the linear space of $m \times n$ matrices over a field $\mathbb{F}$. Then a linear subspace $\mathcal{S} \leq \mathrm{M}(m \times n, \mathbb{F})$ is called a *matrix space*.

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# Basic notation

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- For $n \in \mathbb{N}$, $[n] := \{1, 2, \ldots, n\}$.
- For $(i, j) \in [m] \times [n]$, let $\mathrm{E}_{i,j}$ be the *elementary matrix* in $\mathrm{M}(m \times n, \mathbb{F})$ where the $(i, j)$th entry is 1, and the remaining entries are 0.

# What do matrix spaces have to do with graphs?

- For a bipartite graph $G = ([m] \cup [n], E)$ or a directed graph $G = ([n], E)$, the adjacency matrix is defined as

$$A_G := \sum_{(i,j)\in E} \mathrm{E}_{i,j}.$$

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- Similarly, the *graphical matrix space* (over $\mathbb{F}$) corresponding to $G$ is defined as

$$\mathcal{S}_G := \mathrm{span}\{\mathrm{E}_{i,j} \mid (i,j) \in E\}.$$

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# Three types of correspondences

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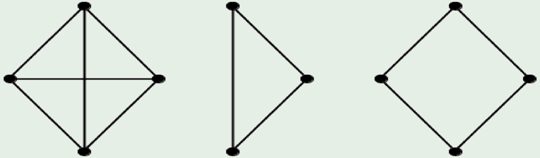
- ***Inherited correspondence***: the maximum number of edges over (spanning) subgraphs of $G$ with property $P$ is equal to the largest dimension over subspaces of $\mathcal{S}_G$ with property $Q$.
- ***Induced correspondence***: the maximum number of vertices over induced subgraphs of $G$ with property $P$ is equal to the order of the maximum induced subspace of $\mathcal{S}_G$ with property $Q$.

# Three types of correspondences

## Example (Induced and Spanning Subgraph)



Original Graph Induced Subgraph Spanning Subgraph

# Matchings in bipartite graphs and ranks of matrices

Basic correspondence:

**Theorem (Edmonds, 1967)**

*Let $G = ([m] \cup [n], E)$ be a bipartite graph with $m \leq n$ and $\mathcal{S}_G \leq \mathrm{M}(m \times n, \mathbb{F})$ be the graphical matrix space associated with $G$. Then for each $r \in [m]$, the matching number of $G$ is at most $r$ if and only if the rank of every matrix in $\mathcal{S}_G$ is at most $r$.*

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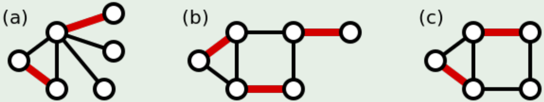
**Example (Matching)**



# Matchings in bipartite graphs and ranks of matrices

Inherited correspondence:

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In general, one of the directions could be easy...

# Thinking more...

Let $m = n$ and $r = n - 1$.

**Corollary**

*Let $G = ([n] \cup [n], E)$ be a bipartite graph and $\mathcal{S}_G \leq \mathrm{M}(n, \mathbb{F})$ be the graphical matrix space associated with $G$. Then the maximum size over subgraphs of $G$ whose matching number is at most $(n-1)$ is equal to the largest dimension over subspaces of $\mathcal{S}_G$ in which every matrix is of rank at most $(n-1)$.*

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# Thinking more...

- If every matrix in a matrix space $\mathcal{S} \leq \mathrm{M}(n, \mathbb{F})$ is singular, what would the upper bound of $\dim(\mathcal{S})$ be?
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**Corollary (Generalisation of Dieudonné's theorem)**

*If every matrix in a matrix space $\mathcal{S} \leq \mathrm{M}(n, \mathbb{F})$ is singular, then*

$$\dim \mathcal{S} \leq n(n-1).$$

# Matchings in bipartite graphs and ranks of matrices

Suppose we have $\mathcal{S} \leq \mathrm{M}(m \times n, \mathbb{F})$, $L \leq \mathbb{F}^m$, $R \leq \mathbb{F}^n$, $\dim(L) = s$ and $\dim(R) = t$. The *order* of $\mathcal{S}$ is $m + n$. Let $T_L$ (resp. $T_R$) be an $s \times m$ (resp. $t \times n$) matrix whose rows span $L$ (resp. $R$). Define $\mathcal{S}[L, R] := \{ T_L B T_R^t \mid B \in \mathcal{S} \}$ be the induced subspace of $\mathcal{S}$ with respect to $L$ and $R$.

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Induced correspondence:

**Theorem**

*Let $G = ([m] \cup [n], E)$ be a bipartite graph with $m \leq n$ and $\mathcal{S}_G \leq \mathrm{M}(m \times n, \mathbb{F})$ be the graphical matrix space associated with $G$. Then for each $r \in [m]$, the maximum order over induced subgraphs of $G$ whose matching number is at most $r$ is equal to the maximum order over induced subspaces of $\mathcal{S}_G$ in which every matrix is of rank at most $r$.*

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# Cycles in directed graphs and nilpotent matrices

**Definition (Nilpotent matrix)**

A matrix $B \in \mathrm{M}(n, \mathbb{F})$ is nilpotent, if $B^k = 0$ for some $k \in \mathbb{N}$.

**Definition (Nil matrix space)**

A matrix space $\mathcal{S} \leq \mathrm{M}(n, \mathbb{F})$ is *nil*, if any $B \in \mathcal{S}$ is a nilpotent matrix.

# Cycles in directed graphs and nilpotent matrices

Basic correspondence:

**Theorem**

*Let $G = ([n], E)$ be a directed graph and $\mathcal{S}_G \leq \mathrm{M}(n, \mathbb{F})$ be the graphical matrix space associated with $G$. Then $G$ is acyclic if and only if $\mathcal{S}_G$ is nil.*

# Cycles in directed graphs and nilpotent matrices

Inherited correspondence:

**Theorem**

*Let* $G = ([n], E)$ *be a directed graph and* $\mathcal{S}_G \leq \mathrm{M}(n, \mathbb{F})$ *be the graphical matrix space associated with* $G$*. Then the maximum size over acyclic subgraphs of* $G$ *is equal to the largest dimension over nil subspaces of* $\mathcal{S}_G$*.*

# Thinking more again...

- If every matrix in a matrix space $\mathcal{S} \leq \mathrm{M}(n, \mathbb{F})$ is nilpotent, what would the upper bound of $\dim(\mathcal{S})$ be?
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# Thinking more again...

- If every matrix in a matrix space $\mathcal{S} \leq \mathrm{M}(n, \mathbb{F})$ is nilpotent, what would the upper bound of $\dim(\mathcal{S})$ be?
- It is equivalent to check the maximum size over acyclic subgraphs of $K_n$ (the complete graph of order $n$)!

**Corollary (Generalisation of Gerstenhaber's theorem)**

*If every matrix in a matrix space $\mathcal{S} \leq \mathrm{M}(n, \mathbb{F})$ is nilpotent, then*

$$\dim \mathcal{S} \leq \frac{n(n-1)}{2}.$$

# Cycles in directed graphs and nilpotent matrices

Let $\mathcal{S} \leq \mathrm{M}(n, \mathbb{C})$ be a matrix space over the complex field $\mathbb{C}$. For a subspace $U \leq \mathbb{C}^n$ of dimension $d$, let $T_U$ be an $d \times n$ matrix whose rows form an orthonormal basis of $U$. The *induced subspace* of $\mathcal{S}$ on $U$ is defined as $\mathcal{S}[U] := \{ T_U B T_U^* \mid B \in \mathcal{S}\} \leq \mathrm{M}(d, \mathbb{C})$.

Induced correspondence:

**Theorem**

*Let $G = ([n], E)$ be a directed graph and $\mathcal{S}_G \leq \mathrm{M}(n, \mathbb{F})$ be the graphical matrix space associated with $G$. Then the maximum order over acyclic subgraphs of $G$ is equal to the largest dimension over $U \leq \mathbb{F}^n$ such that $\mathcal{S}_G[U]$ is nil.*

# Cycle covers and the number of zero eigenvalues

**Definition ($r$-acyclic graph)**

Let $G$ be a directed graph of order $n$, and $r \in \{0\} \cup [n]$. We say $G$ is *$r$-acyclic* if any collection of vertex-disjoint cycles of $G$ covers at most $n - r$ vertices.

Note that (1) $r = 0$ corresponds to $G$ having a cycle cover; and (2) $r = n$ corresponds to $G$ being acyclic.

# Cycle covers and the number of zero eigenvalues

Basic correspondence:

**Theorem**

*Let $G = ([n], E)$ be a directed graph and $\mathcal{S}_G \leq \mathrm{M}(n, \mathbb{F})$ be the graphical matrix space associated with $G$. Then $G$ is $r$-acyclic if and only if every matrix $B \in \mathcal{S}_G$ has at least $r$ zero eigenvalues.*

# Cycle covers and the number of zero eigenvalues

Inherited correspondence:

**Conjecture**

*Let $G = ([n], E)$ be a directed graph and $\mathcal{S}_G \leq \mathrm{M}(n, \mathbb{F})$ be the graphical matrix space associated with $G$. Then for each $r \in \{0\} \cup [n]$, the maximum size over $r$-acyclic subgraphs of $G$ is equal to the largest dimension over subspaces of $\mathcal{S}_G$ in which every matrix has at least $r$ zero eigenvalues.*

# Cycle covers and the number of zero eigenvalues

If the conjecture is true, we can further get

**Corollary (Generalisation of Atkinson's theorem)**

*If every matrix in a matrix space* $\mathcal{S} \leq \mathrm{M}(n, \mathbb{F})$ *has at most* $k$ *non-zero eigenvalues, then*

$$\dim \mathcal{S} \leq nk + \binom{n-k}{2}.$$

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# At the end...

- More conncections are found between strong connectivity and irreducibility, between isomorphism and conjugacy/congruence and so on...
- We also showed some implications of our results to quantum information.
- For more details, please refer to our preprint [arXiv:2206.04815].

Thank you so much!