The stochastic sandpile model on complete graphs

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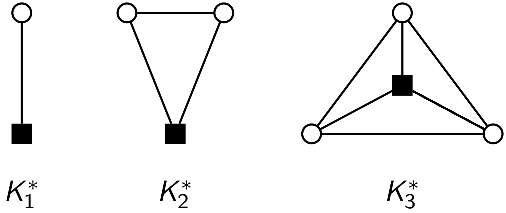
Introduction: the sandpile model

- Sandpile model: grains of sand move around on a graph.
- Introduced by Bak, Tang and Wiesenfeld in 1987. Generalised and formalised by Deepak Dhar in 1990.
  - Exhibits phenomenon of "self-organised criticality".
- Rich research topic in Mathematics, Computer Science (chip-firing game), Statistical Physics.
- This talk focuses on a stochastic variant of the sandpile model (SSM) on complete graphs.

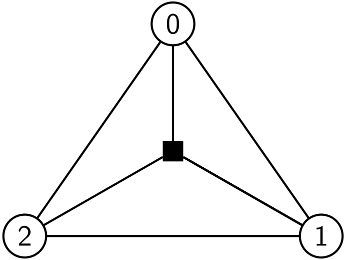
Complete (sandpile) graphs



- In sandpile model, a distringuished vertex called the *sink*.
- Notation: $K_n^* =$ complete sandpile graph on $(n+1)$ vertices (n vertices and the sink). Non-sink vertices are labelled $1, \cdots, n$.

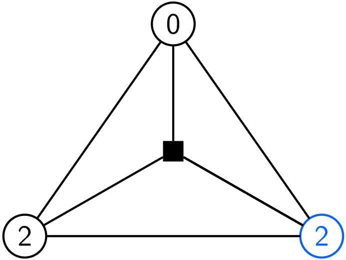
Configurations

- A *configuration* is a vector $c = (c_1, \cdots, c_n) \in \mathbb{Z}_+^n$, where $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$.
- $c_i \equiv$ number of "grains of sand" at vertex i.
- $\mathrm{Config}\,(K_n^*) := \{\text{configurations on } K_n^*\}$.



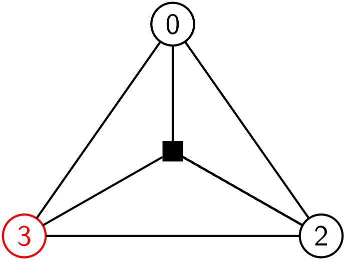
Dynamics (1)

- Let μ be a probability distribution with support $[n] = \{1, \cdots, n\}$.
- At each unit of time, pick a vertex $i \in [n]$ according to μ and add a grain to i, i.e. $c \to c + \mathbb{1}_i$.



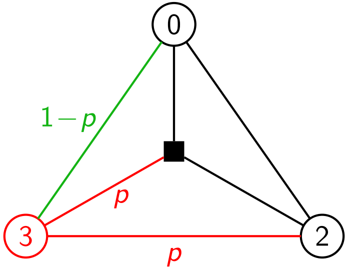
Dynamics (2)

- i is *stable* if $c_i < n = \deg_i$, *unstable* otherwise.
- c is stable if all vertices are stable.
- $\mathrm{Stable}(K_n^*) := \{$stable configurations on $K_n^*\}$.



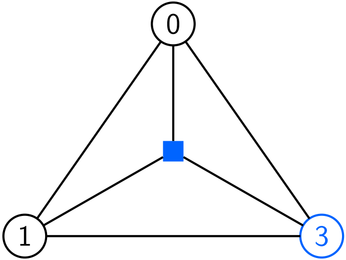
Dynamics (3)

- Unstable vertices *topple* (fire/collapse).
- In SSM, topplings are *random*. For each incident edge, flip a biased (independent) coin:
  - with probability p, send one grain along that edge;
  - with probability $1-p$, keep the grain.



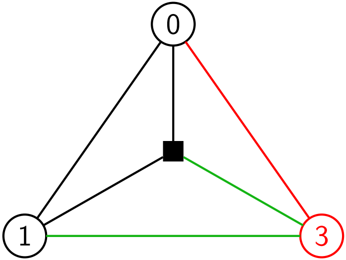
Dynamics (4)

- Unstable vertices *topple* (fire/collapse): grains sent to a p-random subset of neighbours.
- The sink absorbs grains.



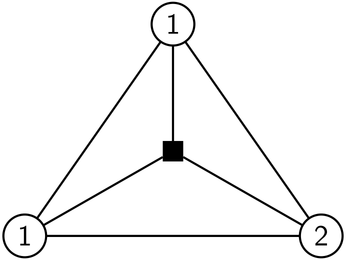
Dynamics (5)

- Toppling one vertex may cause other vertices to become unstable.
- Topple these in turn (independently from previous topplings).



Dynamics (6)

- Aftter finitely many topplings, reach a stable configuration.
- Stabilisation is (can be made) independent of order of topplings.
- $\mathrm{RS}(c) :=$ (random) stabilisation of c.



Recurrent configurations

Stochastic sandpile model (SSM):

- Define a Markov chain on $\mathrm{Stable}\left(K_n^*\right)$: $c \to \mathrm{RS}\left(c + \mathbb{1}_i\right)$, where i is picked according to distribution μ.
- Markov chain eventually becomes trapped in set of stochastically recurrent (SR) configurations $\mathrm{StoRec}\left(K_n^*\right) \subseteq \mathrm{Stable}\left(K_n^*\right)$.
  - These configurations appear infinitely often in long-time running.
- One of a number of SSM in the literature.

Deterministic sandpile model (DSM):

- Take $p = 1$: topplings send one grain to all neighbours.
- Deterministic stabilisation. Randomness in Markov chain is only in μ.
- Set of deterministically recurrent (DR) configurations $\mathrm{DetRec}\left(K_n^*\right) \subseteq \mathrm{Stable}\left(K_n^*\right)$.
- Called *Abelian* sandpile model in literature.

Dhar's burning algorithm

Theorem (Dhar 90)

Let $c \in \mathrm{Stable}(K_n^*)$. Then c is DR iff $\mathrm{DS}\Big(c + (1, \cdots, 1)\Big) = c$. Moreover, in the stabilisation of $c + (1, \cdots, 1)$, every vertex $i \in [n]$ topples exactly once.

Relatively straightforward to check if a given configuration is recurrent or not.

Question

How to calculate $\mathrm{DetRec}(K_n^*)$?

Recurrent configurations and parking functions

Definition (Konheim, Weiss 66)

Let $a = (a_1, \cdots, a_n) \in \mathbb{N}^n$. We say that a is a parking function if its increasing rearrangement $b = (b_1, \cdots, b_n)$ satisfies $b_i \leq i$ for all $i \in [n]$.

Theorem (Cori, Rossin 00)

Let $c \in \mathrm{Stable}(K_n^*)$. Then c is DR iff $n - c := (n - c_1, \cdots, n - c_n)$ is a parking function.
The map $c \mapsto n - c$ is a bijection $\mathrm{DetRec}(K_n^*) \to \mathrm{PF}_n$.

Since then, a rich literature on combinatorial aspects of ASM on various graph families (complete bipartite graphs, wheel graphs, etc.).

Characterisation of SR states

Theorem (Chan Markert S. 13, S. 22+)

Let $c \in \mathrm{Stable}\,(K_n^*)$ and write $\tilde{c} = (\tilde{c}_1, \cdots, \tilde{c}_n)$ for the increasing rearrangement of c. The following are equivalent.

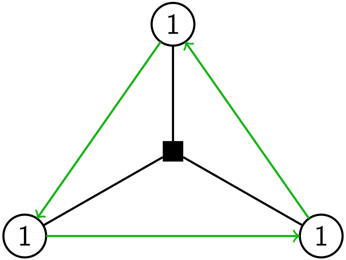
1. $c \in \mathrm{StoRec}\,(K_n^*)$.
2. There exists an orientation $\mathcal{O}$ of K_n (n-tournament) s.t. for all $i \in [n]$, $\mathrm{in}_i^{\mathcal{O}} \leq c_i$.
3. For all $k \in [n]$, we have $\tilde{c}_1 + \cdots + \tilde{c}_k \geq \frac{k(k-1)}{2} = |E(K_k)|$.

Question

How to calculate $\mathrm{StoRec}\,(K_n^*)$?

DR vs SR states

For $n \geq 3$, $\mathrm{DetRec}\,(K_n^*) \subsetneq \mathrm{StoRec}\,(K_n^*)$.



c is SR but not DR ($c + (1, \cdots, 1)$ is stable).

The set $\mathrm{StoRec}\,(K_n^*)$

Theorem (S. 22+)

$\mathrm{StoRec}\,(K_n^*)$ is the set of integer lattice points in the convex hull of $\mathrm{DetRec}\,(K_n^*)$. That is, $c = (c_1, \ldots, c_n) \in \mathrm{Config}\,(K_n^*)$ is SR iff there exists DR configurations $c^{(1)}, \ldots, c^{(k)} \in \mathrm{DetRec}\,(K_n^*)$, and scalars $\lambda_1, \ldots, \lambda_k \in [0, 1]$ with $\sum_{i=1}^{k} \lambda_i = 1$, such that $c = \sum_{i=1}^{k} \lambda_i c^{(i)}$.

Open question

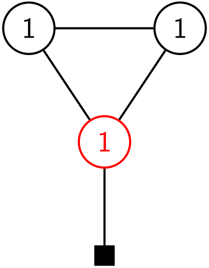
Nice enumerative formula for $\left(\left|\mathrm{StoRec}\,(K_n^*)\,\right|\right)_{n \geq 1}$ = sequence A333331 in OEIS = 1, 3, 17, 144, 1623, 22804, 383415, 7501422, . . . ?

Corollary

The number of integer lattice points in the n-permutation polytope (convex hull of n-permutations S_n) is the number of labelled spanning forests on n vertices.

General graphs

Note: previous result is false on general graphs.



$c(v) = 2$ if $c \in \mathrm{DetRec}(G)$.

Proof sketch

1. Reduction argument: can assume $\max_{i \in [n]} c_i = n - 1$.
   1. If not the case, can write c as convex sum of two SR configurations with higher max.
2. Then straighforward induction.

Open question

More constructive proof?

Partial SSMs (1)

- Define the k-partial SSM to be the sandpile model in which vertices $\{1, \cdots, k\}$ topple stochastically, and $\{k+1, \cdots, n\}$ topple deterministically.
- $\mathrm{PartStoRec}^{(k)}(K_n^*) = \{$recurrent configurations for k-partial SSM$\}$, called k-SR configurations.

Proposition (S. 22+)

For $n \geq 1$, we have:

$$\mathrm{DetRec}(K_n^*) = \mathrm{PartStoRec}^{(0)}(K_n^*) \subseteq \mathrm{PartStoRec}^{(1)}(K_n^*) \subseteq \cdots \subseteq \mathrm{PartStoRec}^{(n)}(K_n^*) = \mathrm{StoRec}(K_n^*).$$

Partial SSMs (2)

Theorem (S. 22+)

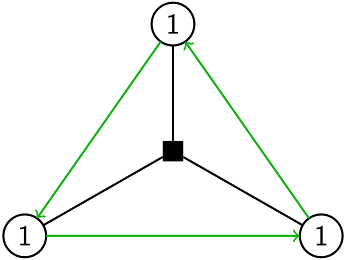
Fix $n \geq 3$. Then we have:

- $\mathrm{DetRec}\left(K_n^*\right) \subsetneq \mathrm{PartStoRec}^{(1)}\left(K_n^*\right)$;
- $\mathrm{PartStoRec}^{(n-3)}\left(K_n^*\right) \subsetneq \mathrm{PartStoRec}^{(n-2)}\left(K_n^*\right) = \mathrm{StoRec}\left(K_n^*\right)$.

Open question

What can we say about the sequences $\left(\left|\mathrm{PartStoRec}^{(k)}\left(K_n^*\right)\right|\right)$?

Idea of proof



- Recall: c is SR but not DR ($c + (1, \cdots, 1)$ is stable).
- This is an example of a "forced cycle" which are key.
  - Forced cycles need stochastic topplings.
  - Don't need many.

Thank you

Thank you for your attention!