Engaging students in combinatorics research

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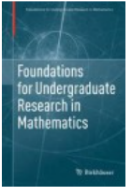
December 13, 2022

My Context (Valparaiso University)

My Context (VERUM)

Resources for student project ideas

- Attend talks, read papers and books



Resources for student project ideas

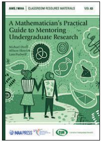
- Attend talks, read papers and books

- Change a variable in an existing problem

Characteristics of good projects for students

- Limited background required
- Specific and concrete
- Multiple layers, simple to more difficult
- Of interest
- Accessible examples (by hand and/or computation)
- You have some idea how to solve it

Characteristics taken from



Three combinatorial stories

- Patterns in words (VERUM 2014, 2015, 2018)
- Patterns in trees (VERUM 2010, 2011, 2012)
- Patterns to the extreme (MathPath 2022)

Permutations

Definition

A permutation is a list where order matters.
$\mathcal{S}_n$ is the set of all permutations of $\{1, 2, \ldots, n\}$.

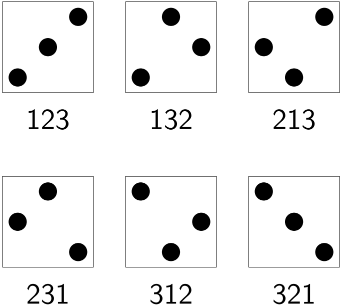
Examples:

- $\mathcal{S}_1 = \{1\}$
- $\mathcal{S}_2 = \{12, 21\}$
- $\mathcal{S}_3 = \{123, 132, 213, 231, 312, 321\}$

$$|\mathcal{S}_n| = n!$$

Plots

Visualize $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n$ by plotting the points (i, π_i) in the xy-plane.



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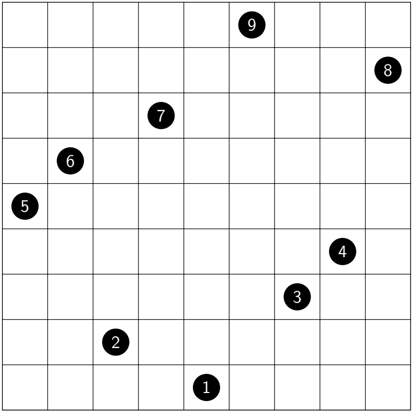
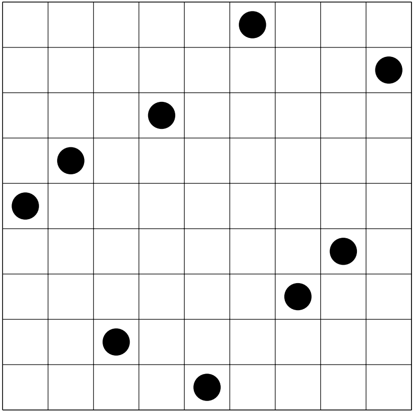
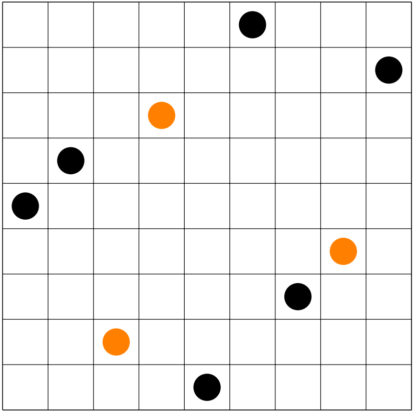


$$\pi = 562719348$$

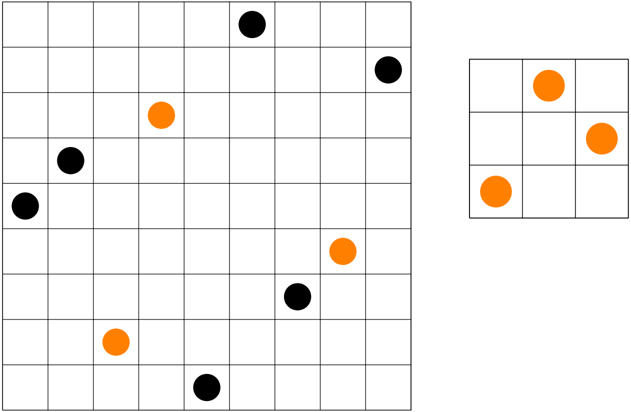
Patterns



Patterns

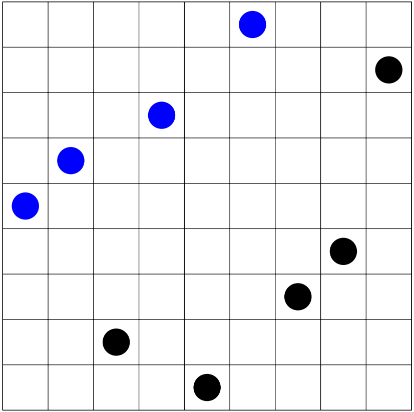


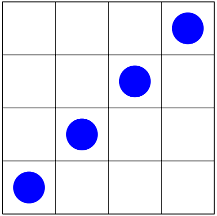
Patterns



562719348 contains the pattern 132

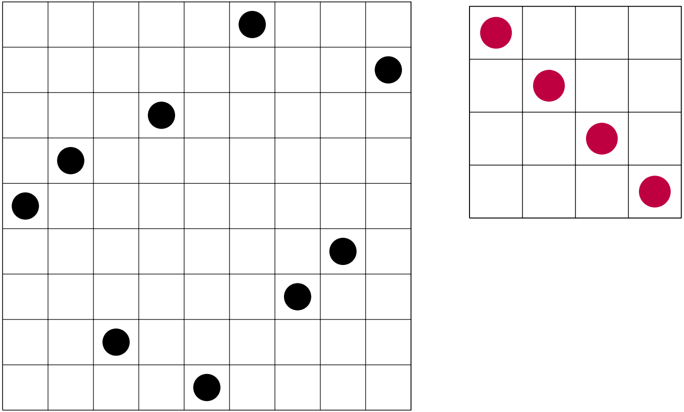
Patterns





562719348 contains the pattern 1234

Patterns



562719348 avoids the pattern 4321

Enumeration

Big question

How many permutations of length n contain the pattern ρ?

Or, alternatively...

Big question

How many permutations of length n avoid the pattern ρ?

(depends on what ρ is!)

Notation

$\mathcal{S}_n(\rho)$ is the set of permutations of length n *avoiding* ρ.
$\mathrm{s}_n(\rho) = |\mathcal{S}_n(\rho)|$.

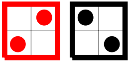
Warm Up

Question

How many permutations of length n avoid the pattern ?

Length 1? (1)

Length 2? (1)



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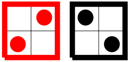
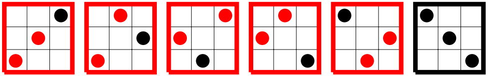


Length 3? (1)



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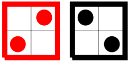

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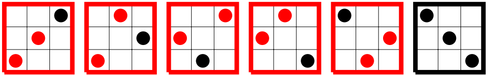


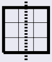
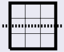
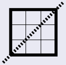
The decreasing permutation is the only permutation of length n that avoids 12.

Similar: the increasing permutation is the only permutation of length n that avoids 21.

Symmetry

Common symmetries

Given $\rho = \rho_1 \cdots \rho_n \in \mathcal{S}_n$,

- $\rho^r = \rho_n \cdots \rho_1$ (reverse) 
- $\rho^c = (n+1-\rho_1)(n+1-\rho_2)\cdots(n+1-\rho_n)$ (complement) 
- ρ^{-1} is graphed by plotting (ρ_i, i) for $1 \leq i \leq n$ (inverse) 

Fact: For any ρ, $\mathrm{s}_n(\rho) = \mathrm{s}_n(\rho^r) = \mathrm{s}_n(\rho^c) = \mathrm{s}_n(\rho^{-1})$.

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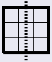
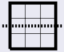
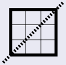
- $\rho^r = \rho_n \cdots \rho_1$ (reverse) 
- $\rho^c = (n+1-\rho_1)(n+1-\rho_2)\cdots(n+1-\rho_n)$ (complement) 
- ρ^{-1} is graphed by plotting (ρ_i, i) for $1 \leq i \leq n$ (inverse) 

Fact: For any ρ, $\mathrm{s}_n(\rho) = \mathrm{s}_n(\rho^r) = \mathrm{s}_n(\rho^c) = \mathrm{s}_n(\rho^{-1})$.

So... $\mathrm{s}_n(12) = \mathrm{s}_n(21)$

$\mathrm{s}_n(123) = \mathrm{s}_n(321)$; $\mathrm{s}_n(132) = \mathrm{s}_n(231) = \mathrm{s}_n(213) = \mathrm{s}_n(312)$

Results

How many permutations of length n avoid the pattern...

- 12? (1)
- 123? (Catalan)
- 132? (Catalan)

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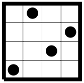
How many permutations of length n avoid the pattern...

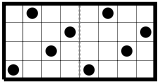
- 12? (1)
- 123? (Catalan)
- 132? (Catalan)
- 1234? $1, 1, 2, 6, 23, 103, 513, 2761, \ldots$ (Gessel, 1990)
- 1342? $1, 1, 2, 6, 23, 103, 512, 2740, \ldots$ (Bóna, 1997)
- 1324? $1, 1, 2, 6, 23, 103, 513, 2762, \ldots$ (open question!)

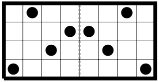
Symmetric Words

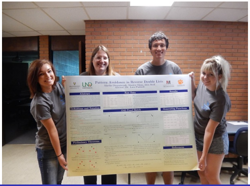
Generalizations of Permutations

$$\mathcal{D}_n = \{\pi\pi | \pi \in \mathcal{S}_n\}$$
$$\mathcal{R}_n = \{\pi\pi^r | \pi \in \mathcal{S}_n\}$$

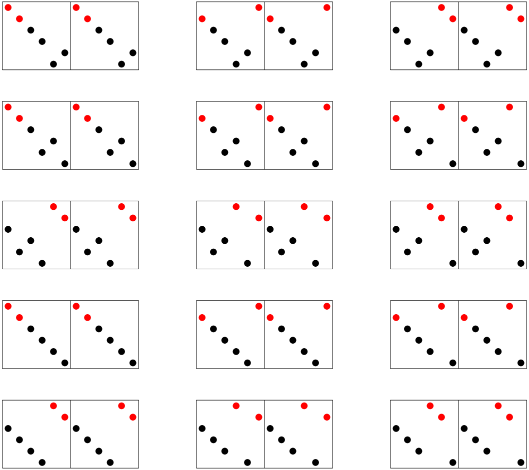
$1423 \in \mathcal{S}_4$ , so

$14231423 \in \mathcal{D}_4$ 

$14233241 \in \mathcal{R}_4$ 



$\mathcal{D}_6(1342)$



Proof Sketch

Let $\pi\pi \in \mathcal{D}_n(1342)$.
Erase n and $n-1$ to obtain $\pi'\pi' \in \mathcal{D}_{n-2}(1342)$.

Observations:

- π' has at most one 12 pattern.

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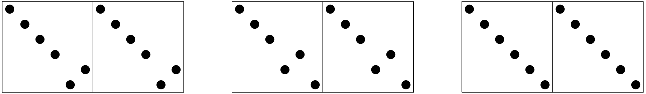
Observations:

- π' has at most one 12 pattern.
- If π' has a 12 pattern, it uses consecutive digits in adjacent positions.
- If π' has a 12 pattern it uses the digits 12 or 23.

3 possible choices for $\pi'\pi'$:



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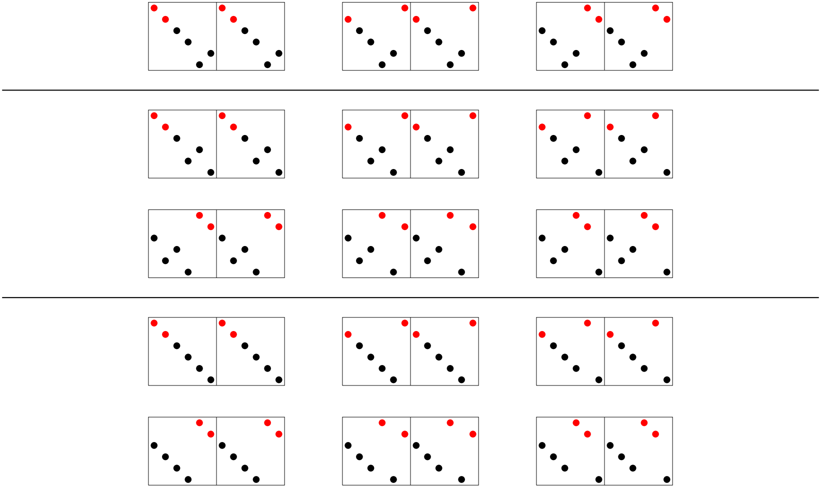

Pattern avoidance in $\mathcal{D}_n$

Pattern ρ	$\mathrm{d}_n(\rho)$	
1342, 2431, 3124, 4213	15	$(n \geq 5)$
2143, 3412	$2n+2$	$(n \geq 6)$
1423, 2314, 3241, 4132	$3n+6$	$(n \geq 7)$
1432, 2341, 3214, 4123	$\frac{1}{2}n^2 + \frac{3}{2}n - 4$	$(n \geq 6)$
1243, 2134, 3421, 4312	$\frac{1}{2}n^2 + \frac{5}{2}n - 8$	$(n \geq 6)$
2413, 3142	L_{n+2}	$(n \geq 5)$
1324, 4231	$\mathrm{d}_{n-1}(\rho) + \mathrm{d}_{n-2}(\rho) + \mathrm{d}_{n-3}(\rho)$	$(n \geq 10)$
1234, 4321	$2^n - n$	$(n \geq 4)$

Pattern avoidance in $\mathcal{R}_n$

Pattern ρ	$\mathrm{r}_n(\rho)$	
1234	0	$(n \geq 7)$
1243	$\frac{n^3}{3} - \frac{7n}{3} + 4$	$(n \geq 3)$
1324 2143	$2\mathrm{r}_{n-1}(\rho) + 4$	$(n \geq 4)$
1423	$2\mathrm{r}_{n-1}(\rho) + \mathrm{r}_{n-3}(\rho) + 2$	$(n \geq 5)$
1432	$2\mathrm{r}_{n-1}(\rho) + \mathrm{r}_{n-2}(\rho)$	$(n \geq 5)$
1342	$2\mathrm{r}_{n-1}(\rho) + \mathrm{r}_{n-2}(\rho) + 2$	$(n \geq 4)$
2413	$2\mathrm{r}_{n-1}(\rho) + 2\mathrm{r}_{n-2}(\rho)$	$(n \geq 3)$

Optimization

Question

What is the max number of copies of ρ in a member of $\mathcal{D}_n$? $\mathcal{R}_n$?

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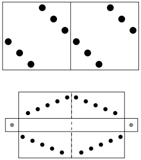
Maximizing copies of 2143



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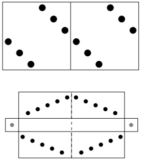
Maximizing copies of 2143



- In $\mathcal{D}_n$ appears to be correlated to copies in $\mathcal{S}_n$.
- In $\mathcal{R}_n$ appears to be sparser than copies in $\mathcal{S}_n$; more open cases.
  - Known: 1234, 1243, 1342, 1432, 2143
  - Open: 1423, 1324, 2413

Other generalizations?

- Avoiding longer patterns
- Avoiding multiple patterns simultaneously
- Pattern avoidance in $\{\pi\pi^c|\pi \in \mathcal{S}_n\}$
- Pattern avoidance in $\left\{\pi\pi^{-1}\middle|\pi \in \mathcal{S}_n\right\}$
- Packing as many copies of a pattern as possible into other structured words.

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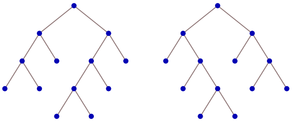
Patterns in Trees

Our trees are:

- rooted (root vertex at top, children below)
- ordered (left child and right child are distinct)
- full binary (each vertex has exactly 0 or 2 children)



$\mathbb{T}_n$ is the set of n-leaf binary trees.

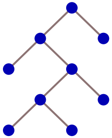
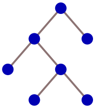
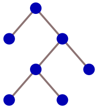
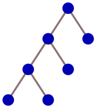
Question: How many trees in $\mathbb{T}_n$ avoid a given tree pattern?

Tree patterns

Contiguous tree pattern

Tree T contains tree t if and only if t is a contiguous rooted ordered subtree of T.

Example:

 contains  and  but avoids .

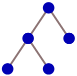
What is the number $a_t(n)$ of n-leaf binary trees avoiding t?

$t =$ 

$a_t(n) = 0$

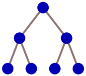
$t =$ 

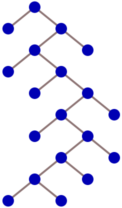
$$a_t(n) = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$$

$t =$ 

$a_t(n) = 1$

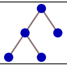
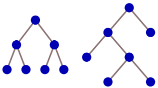
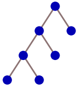
What is the number $a_t(n)$ of n-leaf binary trees avoiding t?

$t =$ 

"Typical" tree avoiding t: 

$$a_t(n) = \begin{cases} 1 & n = 1 \\ 2^{n-2} & n > 1 \end{cases}$$

Contiguous pattern enumeration data

t	$a_t(n)$
	0
	$\begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$
	1
	2^{n-2}
	M_{n-1} (Motzkin numbers)

Contiguous tree pattern results

- Rowland: contiguous pattern avoidance in binary trees
- Gabriel, Peske, P., Tay: extended Rowland's results to ternary trees

Contiguous tree pattern results

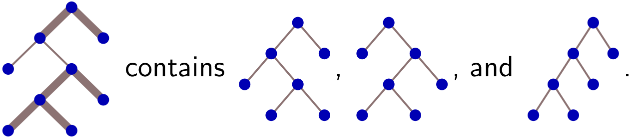
- Rowland: contiguous pattern avoidance in binary trees
  - Algorithm to determine $\sum_{n\geq 1} a_t(n)x^n$ for any binary tree pattern.
  - $\sum_{n\geq 1} a_t(n)x^n$ is always algebraic.
- Gabriel, Peske, P., Tay: extended Rowland's results to ternary trees
  - Counting results from avoiding ternary tree patterns include Catalan numbers, little Schröder numbers, and other known combinatorial sequences.

Tree patterns

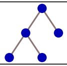
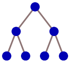
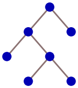
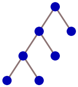
Noncontiguous tree pattern

Tree T contains tree t if and only if there exists a sequence of edge contractions of T (by pairs) that produces t.

Example:



Noncontiguous pattern enumeration data

Pattern t	Number of n-leaf trees avoiding t
	0
	$\begin{cases} 1 & n=1 \\ 0 & n>1 \end{cases}$
	1
  	2^{n-2}

The Main Theorem

Notation

- Let $\mathrm{av}_t(n)$ be the number trees in $\mathbb{T}_n$ that avoid t noncontiguously.
- Let $g_t(x) = \sum_{n=1}^{\infty} \mathrm{av}_t(n) x^n$.

Theorem (Dairyko, P., Tyner, & Wynn, 2011)

Fix $k \in \mathbb{Z}^+$. Let $t, s \in \mathbb{T}_k$. Then $g_t(x) = g_s(x)$.

Generating functions

k	$g_t(x)$, $t \in \mathbb{T}_k$	OEIS number
1	0	trivial
2	x	trivial
3	$\frac{x}{1-x}$	trivial
4	$\frac{x-x^2}{1-2x}$	A000079
5	$\frac{x-2x^2}{1-3x+x^2}$	A001519
6	$\frac{x-3x^2+x^3}{1-4x+3x^2}$	A007051
7	$\frac{x-4x^2+3x^3}{1-5x+6x^2-x^3}$	A080937
8	$\frac{x-5x^2+6x^3-x^4}{1-6x+10x^2-4x^3}$	A024175

Coefficient sightings...

$$\frac{x}{1}$$

$$\frac{x}{1-x}$$

$$\frac{x-x^2}{1-2x}$$

$$\frac{x-2x^2}{1-3x+x^2}$$

$$\frac{x-3x^2+x^3}{1-4x+3x^2}$$

$$\frac{x-4x^2+3x^3}{1-5x+6x^2-x^3}$$

$$\frac{x-5x^2+6x^3-x^4}{1-6x+10x^2-4x^3}$$

Coefficient sightings...

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

$$\frac{x}{1}$$

$$\frac{x}{1-x}$$

$$\frac{x-x^2}{1-2x}$$

$$\frac{x-2x^2}{1-3x+x^2}$$

$$\frac{x-3x^2+x^3}{1-4x+3x^2}$$

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Coefficient sightings...

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

$$\frac{x}{1}$$

$$\frac{x}{1-x}$$

$$\frac{x-x^2}{1-2x}$$

$$\frac{x-2x^2}{1-3x+x^2}$$

$$\frac{x-3x^2+x^3}{1-4x+3x^2}$$

$$\frac{x-4x^2+3x^3}{1-5x+6x^2-x^3}$$

$$\frac{x-5x^2+6x^3-x^4}{1-6x+10x^2-4x^3}$$

An explicit formula

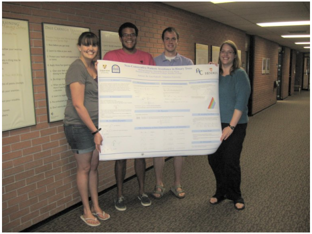
Theorem (Dairyko, P., Tyner, & Wynn, 2011)

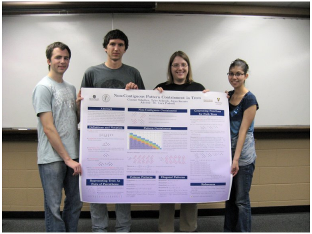
Let $k \in \mathbb{Z}^+$ and let $t \in \mathbb{T}_k$. Then

$$g_t(x) = \frac{\displaystyle\sum_{i=0}^{\lfloor \frac{k-2}{2} \rfloor} (-1)^i \cdot \binom{k-(i+2)}{i} \cdot x^{i+1}}{\displaystyle\sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^i \cdot \binom{k-(i+1)}{i} \cdot x^i}.$$

Noncontiguous tree pattern results

- Dairyko, P., Tyner, Wynn: **noncontiguous** pattern avoidance in binary trees
- P., Serrato, Scholten, Schrock: noncontiguous pattern **containment** in *m*-ary trees





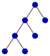
Contiguous Containment

Theorem (Flajolet & Steyaert, 1983)

The number of copies of a k-leaf tree in the set of all n-leaf trees is independent of the tree pattern and is $\binom{2n-k}{n-k}$.

Example: Any 4-leaf tree is contained in the set of 5-leaf trees $\binom{2\cdot 5-4}{5-4} = \binom{6}{1} = 6$ times.

Contiguous Containment Example

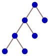
6 copies of  in 5-leaf trees

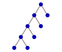
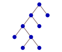
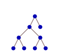
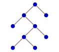
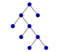
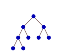
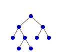
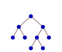
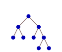
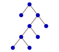
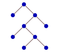
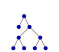
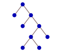
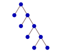
2	1	1	0	0	1	0
0	0	1	0	0	0	0

6 copies of  in 5-leaf trees

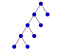
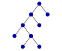
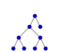
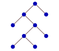
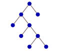
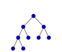
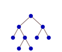
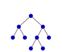
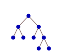
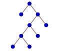
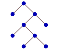
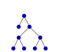
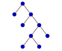
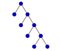
0	0	1	0	0	1	1
1	1	0	0	1	0	0

Noncontiguous Containment Example

10 copies of  in 5-leaf trees

						
4	2	1	1	0	1	0
						
0	0	1	0	0	0	0

10 copies of  in 5-leaf trees

						
0	0	1	0	0	2	2
						
2	2	0	0	1	0	0

Noncontiguous Containment

Theorem (P., Scholten, Schrock, Serrato, 2012)

If $\mathrm{occ}_k(n)$ is the number of noncontiguous occurrences of $t \in \mathbb{T}_k$ in $\mathbb{T}_n$, then $\mathrm{occ}_k(n)$ is independent of t and

$$\sum_{n\geq 1}\sum_{k\geq 1}\mathrm{occ}_k(n)x^n y^k = \frac{\sqrt{1-4x}(1-\sqrt{1-4x})y}{(y+2)\sqrt{1-4x}-y}.$$

Expansion:

$$\frac{\sqrt{1-4x}(1-\sqrt{1-4x})y}{(y+2)\sqrt{1-4x}-y} = xy + x^2(y+y^2) + x^3(2y+4y^2+y^3)+$$

$$x^4(5y+15y^2+7y^3+y^4)+x^5(14y+56y^2+37y^3+\mathbf{10}y^4+y^5)+\cdots$$

Recap

- Rowland: How many *binary* trees *avoid* a given *contiguous* tree pattern?
- Variation 1: How many *ternary* trees *avoid* a given *contiguous* tree pattern?
- Variation 2: How many *binary* trees *avoid* a given *noncontiguous* tree pattern?
- Variation 3: How many *m-ary* trees *contain* a given *noncontiguous* tree pattern?

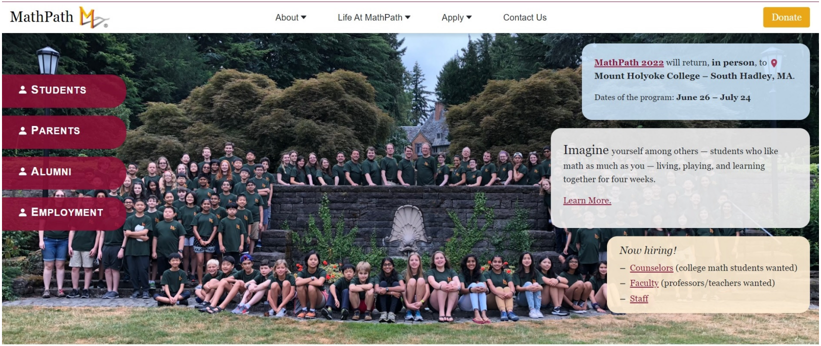
Other generalizations?

- non-ordered trees
- non-rooted trees
- non-full trees
- semi-contiguous tree patterns
  (some parts must be contiguous; other parts not)

Characteristics of good projects for students

- Limited background required ✓
- Specific and concrete ✓
- Multiple layers, simple to more difficult ✓
- Of interest ✓
- Accessible examples (by hand and/or computation) ✓
- You have some idea how to solve it ✓

MathPath



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Student work

ISOPERIMETRIC SETS OF ENGLISH WORDS

GB,TC,MC,MD,BF,WG,XH,SK,TL,PM,NM,AN,EP,ZT,AU,EZ

ABSTRACT. We consider a space of English words, define distance and boundary, and find subsets of small volumes of minimum or maximum perimeter.

1. INTRODUCTION

The isoperimetric problem is among the oldest in mathematics. It asks for the least-perimeter way to enclose a given volume. In this paper, we consider a space of English words and seek subsets of small volumes of minimum or maximum perimeter.

While there has been much study of numerical properties of the English language [4] and of the isoperimetric problem on discrete spaces (see e.g. [2] and [1]), as far as we know, our specific focus is new.

We use a standard list of the 3000 most common English words [3], with a notion of distance and perimeter (Sect. 2). We begin with the study of singletons (word sets with volume 1).

Classic result

Theorem (Erdős-Szekeres, 1935)

Any permutation of length $(a-1)(b-1)+1$ or more contains either an increasing pattern of length a or a decreasing pattern of length b.

Classic result

Theorem (Erdős-Szekeres, 1935)

Any permutation of length $(a-1)(b-1)+1$ or more contains either an increasing pattern of length a or a decreasing pattern of length b.

Examples: $a = 3,\ b = 4$

- 321654 avoids 123 and 4321.
- 563412 avoids 123 and 4321.
- 3216547 contains 367.
- 1726354 contains 123 and 7654.

The (a, b)-permutation game

- Take turns naming different positive numbers.
- The game is over when a player completes an increasing sequence of length a or a decreasing sequence of length b.

Example for the $(3, 3)$-permutation game:
Player 1 picks 5
Player 2 picks 1
Player 1 picks 4
Player 2 picks 2
The game is over because 5,4,2 is a decreasing sequence of length 3.

Question: How long can you make the game last?

The (a, b)-permutation game

- The longest $(3, 3)$-game is ____ moves.
- The longest $(3, 4)$-game is ____ moves.
- The longest $(3, 5)$-game is ____ moves.
- The longest $(4, 4)$-game is _____ moves.
- The longest $(4, 5)$-game is _____ moves.

The (a, b)-permutation game

- The longest $(3,3)$-game is 5 moves.
- The longest $(3,4)$-game is 7 moves.
- The longest $(3,5)$-game is 9 moves.
- The longest $(4,4)$-game is 10 moves.
- The longest $(4,5)$-game is 13 moves.

Fact: The longest (a, b)-game is $(a-1)(b-1)+1$ moves.

A detour

One strategy:

A detour

One strategy:

- (avoiding 123 and 4321): 321654
- (avoiding 1234 and 4321): 321654987

A detour

One strategy:

- (avoiding 123 and 4321): 321654
- (avoiding 1234 and 4321): 321654987

Question

How many different permutations of length $(a-1)(b-1)+1$ result from playing the (a, b)-permutation game optimally?

Student-inspired observation

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Proposition

There are $(C_{a-1})^2$ permutations of length $2(a-1)$ that avoid both $12\cdots a$ and 321.

Why?

Student-inspired observation

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Why?

- Robinson-Schensted-Knuth correspondence

Student-inspired observation

Question

How many different permutations of length $(a-1)(b-1)+1$ result from playing the (a,b)-permutation game optimally?

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There are $(C_{a-1})^2$ permutations of length $2(a-1)$ that avoid both $12\cdots a$ and 321.

Why?

- Robinson-Schensted-Knuth correspondence
- bijection with pairs of parentheses arrangements

Bijection

permutation	left-to-right maxima	positions	parentheses pairs
2143	{2,4}	{1,3}	()(), ()()
2413	{2,4}	{1,2}	()(), (())
3412	{3,4}	{1,2}	(()), (())
3142	{3,4}	{1,3}	(()), ()()

Lessons Learned

- Change a parameter or definition.

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- Be willing to learn new topics.

Lessons Learned

- Change a parameter or definition.
- Be willing to learn new topics.
- Be open to student-inspired lines of questions.

For more details (on permutations, words, and trees)...

Words & Permutations:

- M. Anderson, M. Diepenbroek, L. Pudwell, and A. Stoll, Pattern avoidance in reverse double lists, *Discrete Math. Theor. Comput. Sci.* **19.2** (2018), #13.
- Miklos Bóna, *Combinatorics of Permutations*, Chapman & Hall, 2004.
- C. Cratty, S. Erickson, F. Negassi, and L. Pudwell, Pattern avoidance in double lists, *Involve* **10.3** (2017), 379–398.
- J. Krull, L. Pudwell, E. Redmon, and A. Reimer-Berg, Packing patterns in symmetric words, *Australas. J. Combin.* **84.2** (2022), 238–257.
- L. Pudwell Catalan Numbers and Permutations, to appear in *Mathematics Magazine*.

Trees:

- M. Dairyko, L. Pudwell, S. Tyner, and C. Wynn, Non-contiguous pattern avoidance in binary trees, *Electron. J. Combin.* **19** (3) (2012), P22.
- P. Flajolet and J. M. Steyaert, Patterns and pattern-matching in trees: an analysis. *Info. Control* **58** (1983), 19–58.
- N. Gabriel, K. Peske, L. Pudwell, and S. Tay, Pattern avoidance in ternary trees, *J. Integer Seq.* **15** (2012), 12.1.5.
- L. Pudwell, C. Scholten, T. Schrock, and A. Serrato, Non-contiguous pattern containment in binary trees, *ISRN Combinatorics* vol. 2014, Article ID 316535, 8 pages, 2014.
- E. S. Rowland, Pattern avoidance in binary trees, *J. Combin. Theory, Ser. A* **117** (2010), 741–758.

For more details (on undergraduate research)...

- *Foundations for Undergraduate Research in Mathematics* series, edited by Aaron Wootton (Springer)
- *A Mathematician's Practical Guide to Mentoring Undergraduate Research* by Michael Dorff, Allison Henrich, and Lara Pudwell (MAA Press 2019)
  (Choosing problems, group dynamics, communication, funding, assessment, and more!)
- Australasian Council for Undergraduate Research
  `https://www.acur.org.au/`
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Thanks for listening!