Three-dimensional maps and subgroup growth

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An outline

Question 1

How many free subgroups of given index are there in $\mathbb{Z}_p * \mathbb{Z}_q$?

Question 2

How many “surfaces” can we glue out of a certain amount of p-gons and q-gons?

An outline

Question 1

How many free subgroups of given index are there in $\mathbb{Z}_p * \mathbb{Z}_q$?

Question 2

How many “surfaces” can we glue out of a certain amount of p-gons and q-gons?

Question 3:

Are the above two questions related?

A reformulation

Question 1

How many free subgroups of given index are there in $\mathbb{Z}_p * \mathbb{Z}_q$?

Question 2

How many polygonal subdivisions of surfaces can we create with a certain amount of p-gons and q-gons?

Question 3:

Are the above two questions in some sense "isomorphic"?

More on surfaces ...

In combinatorial/group theoretic language,

what is a combinatorial (orientable) hypermap?

Definition

A hypermap is a couple of permutations $\alpha, \sigma \in \mathfrak{S}_n$, such that $\langle \alpha, \sigma \rangle$ acts transitively on $\{1, 2, 3, \ldots, n\}$.

A bit of geometry

Orbits of σ = vertices, orbits of α = hyper-edges, orbits of $\varphi = \sigma^{-1}\alpha^{-1}$ = hyper-faces.

More on surfaces ...

In topological/geometric language,

what is a geometric (orientable) hypermap?

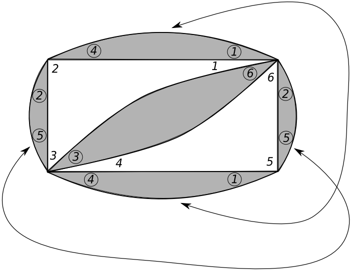
Definition

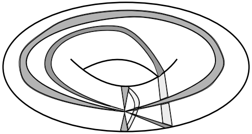
A hypermap is a graph embedded into a genus g orientable surface, such that its complement is a collection of open discs (faces) and its faces are properly two colourable (say, B/W).

A bit of combinatorics

If we label the corners of white faces, we obtain a labelled hypermap.

Example





This is a hypermap on a torus with $\alpha = (1,4)(2,5)(3,6)$, $\sigma = (5,1,6,2,4,3)$, $\varphi = (1,2,3)(5,6,4)$.

A kind of dictionary ...

Two instances of same object:

we can pass from combinatorial to geometric hypermaps and back.

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A rich family of hypermaps

Definition

A (p, q)-hypermap H is a hypermap in which α has only cycles of length p and φ has cycles of length q only. In other words, all hyper-edges of H are p-gons and all hyper-faces of H are q-gons.

Notation

$\mathfrak{H}^r_{p,q}(n)$ = rooted (p, q)-hypermaps on n darts (corner labels for white faces).

Notation again ...

$\mathfrak{H}_{p,q}(n) = (p, q)$-hypermaps on n darts up to isomorphism.

Building a dictionary

Lemma 1

Let p, q be two natural numbers, $pq \geq 6$. There is a one-to-one correspondence between the set of connected oriented rooted (p, q)-hypermaps $\mathfrak{H}^r_{p,q}(n)$ on n darts and the set of free subgroups of index n in the group $\Delta^+ = \mathbb{Z}_p * \mathbb{Z}_q$.

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One more lemma ...

Lemma 2

Let p, q be two natural numbers, $pq \geq 6$. There is a one-to-one correspondence between the set of isomorphism classes of connected oriented (p, q)-hypermaps $\mathfrak{H}_{p,q}(n)$ on n darts and the set of conjugacy classes of free subgroups of index n in the group $\Delta^+ = \mathbb{Z}_p * \mathbb{Z}_q$.

Lemmas 1 and 2

Our results follow Jones-Singerman (who initiated theory of maps on orientable surfaces) and, later on, Breda-Mednykh-Nedela (who solved Tutte's enumeration problem): a transitive finite group action on an n–element set can be described as the action of some group on the set of cosets of some of its index n subgroups.

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For rooted (p, q)-hypermaps ...

Theorem 1

The growth series $H^{\circ}(z) = \sum_{n=0}^{\infty} \mathrm{card}\, \mathfrak{H}^{r}_{p,q}(n) \cdot z^{n}$ can be efficiently computed and the following asymptotic formula holds for its non-zero coefficients:

$$[z^{\langle p,q\rangle k}]H^{\circ}(z) \propto \frac{(2\pi)^{\frac{N-1}{2}} \langle p, q\rangle}{\prod_i \Gamma(a_i)} \; k^{(N-1)k-\frac{N-1}{2}+\sum_i a_i} \; e^{-(N-1-\log c)k}, \text{ as } k \to \infty,$$

where $N = 1 + \frac{(p-1)(q-1)-1}{(p,q)}$, and the constants c and a_i satisfy $c^{(p,q)} = \langle p, q\rangle^{(p-1)(q-1)-1}$, $a_i = \frac{i(p,q)}{pq}$, with $i = 1, \ldots, \frac{pq}{(p,q)} - 1$, $p \nmid i$, $q \nmid i$.

$(\, , \,) =$ g.c.d., $\langle \, , \, \rangle =$ l.c.m.

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For rooted (p, q)-hypermaps ...

The series $H^\circ(z)$ is related to the log of the hypergeometric series

$$_{1+\frac{(p-1)(q-1)-1}{(p,q)}}F_0\left[\begin{array}{c} \frac{i(p,q)}{pq}, i = 1, \ldots, \frac{pq}{(p,q)} - 1, p \nmid i, q \nmid i \\ \ldots \end{array} ; z\right].$$

For unrooted (p,q)-hypermaps ...

Theorem 2

The growth series $\widetilde{H}(z) = \sum_{n=0}^{\infty} \mathrm{card}\, \mathfrak{H}_{p,q}(n) \cdot z^n$ is given by the formula

$$\widetilde{H}(z) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \sum_{k=1}^{\infty} \log(P_n \odot Q_n)(z^{nk}),$$

with

$$P_n(z) = \exp\left(\frac{1}{np} \sum_{d|(n,p)} d\,\phi(d)\, z^{p/d} \right),$$

and

$$Q_n(z) = \exp\left(\frac{1}{nq} \sum_{d|(n,q)} d\,\phi(d)\, z^{q/d} \right).$$

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For unrooted (p, q)-hypermaps ...

Theorem 2 (cont.)

The non-zero coefficients of $\widetilde{H}(z)$ have the following asymptotic expansion:

$$[z^{\langle p,q\rangle k}]\widetilde{H}(z) \propto \frac{(2\pi)^{\frac{N-1}{2}}}{\prod_i \Gamma(a_i)}\ k^{(N-2)k-\frac{N-1}{2}+\sum_i a_i}\ e^{-(N-1-\log c)k},\ \text{as } k \to \infty,$$

where $N = 1 + \frac{(p-1)(q-1)-1}{(p,q)}$, and the constants c and a_i satisfy $c^{(p,q)} = \langle p, q\rangle^{(p-1)(q-1)-1}$, $a_i = \frac{i(p,q)}{pq}$, with $i = 1, \ldots, \frac{pq}{(p,q)} - 1$, $p \nmid i$, $q \nmid i$.

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Some examples

Rooted triangulations

on n darts have generating series $H^{\circ}(z) = \sum_{n=0}^{\infty} \operatorname{card} \mathfrak{H}^{r}_{2,3}(n) \cdot z^{n} = 5z^{6} + 60z^{12} + 1105z^{18} + 27120z^{24} + 828250z^{30} + 30220800z^{36} + 1282031525z^{42} + 61999046400z^{48} + 3366961243750z^{54} + 202903221120000z^{60} + \ldots$.

Its coefficient sequence has number A062980 in the OEIS (by Petitot and Vidal).

Some examples

Rooted quadrangulations

on n darts have generating series $H^{\circ}(z) = \sum_{n=0}^{\infty} \operatorname{card} \mathfrak{H}_{2,4}^{r}(n) \cdot z^n = 3z^4 + 24z^8 + 297z^{12} + 4896z^{16} + 100278z^{20} + 2450304z^{24} + 69533397z^{28} + 2247492096z^{32} + 81528066378z^{36} + +3280382613504z^{40} + \ldots$.

Its coefficient sequence has number A292186 in the OEIS (new!).

Some examples

Rooted bi-coloured triangulations

on n darts have generating series $H^\circ(z) = \sum_{n=0}^{\infty} \operatorname{card} \mathfrak{H}^{r}_{3,3}(n) \cdot z^n = 2z^3 + 12z^6 + 112z^9 + 1392z^{12} + 21472z^{15} + 394752z^{18} + 8421632z^{21} + 204525312z^{24} + 5572091392z^{27} + 168331164672z^{30} + \ldots.$

Its coefficient sequence has number A292187 in the OEIS (new!).

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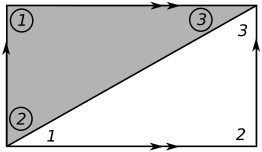
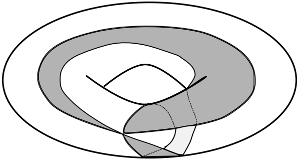
A bi-coloured triangulation





This is a hypermap on a torus with $\alpha = \sigma = \varphi = (1, 2, 3)$.

More examples

Unrooted triangulations

on n darts have generating series $\widetilde{H}(z) = \sum_{n=0}^{\infty} \mathrm{card}\, \mathfrak{H}_{2,3}(n) \cdot z^n = 3z^6 + 11z^{12} + 81z^{18} + 1228z^{24} + 28174z^{30} + 843186z^{36} + 30551755z^{42} + 1291861997z^{48} + 62352938720z^{54} + 3381736322813z^{60} + \ldots$.

Its coefficient sequence has number A129114 in the OEIS (by Petitot and Vidal).

More examples

Unrooted quadrangulations

on n darts have generating function $\widetilde{H}(z) = \sum_{n=0}^{\infty} \mathrm{card}\, \mathfrak{H}_{2,4}(n) \cdot z^n = 2z^4 + 7z^8 + 36z^{12} + 365z^{16} + 5250z^{20} + 103801z^{24} + 2492164z^{28} + 70304018z^{32} + 2265110191z^{36} + 82013270998z^{40} + \dots$.

Its coefficient sequence has number A292206 in the OEIS (new!).

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More examples

Unrooted bi-coloured triangulations

on n darts have generating function $\widetilde{H}(z) = \sum_{n=0}^{\infty} \operatorname{card} \mathfrak{H}_{3,3}(n) \cdot z^n = 2z^3 + 3z^6 + 16z^9 + 133z^{12} + 1440z^{15} + 22076z^{18} + 401200z^{21} + 8523946z^{24} + 206375088z^{27} + 5611089408z^{30} + \ldots$.

Its coefficient sequence has number A292207 in the OEIS (new!).

One dimension higher

A theory of three-dimensional hypermaps (or pavements, or pavings) by Arquès-Koch, Spechner, Lienhardt: study the associated objects from the combinatorial and group-theoretic points of view.

One dimension higher

Suppose that H is a map (i.e. a $(2, q)$-hypermap), not necessarily connected: let $H = H_1 \sqcup \cdots \sqcup H_m$. Such a map H is given by a pair of permutations $\langle \alpha, \sigma \rangle \subset \mathfrak{S}_n$, as before, although not necessarily transitive.

Let $\varphi \in \mathfrak{S}_n$ be a permutation on the darts of H that indicates how the faces of H_i's should be glued together (by identification on their boundary darts), so that the resulting object is a connected oriented topological space (with an additional combinatorial structure of a "handlebody decomposition" into maps).

Paving: definition

A three-dimensional connected oriented rooted map, or paving, can be described by a transitive triple $\langle \alpha, \sigma, \varphi \rangle$ such that

(I-1) the product $\alpha\varphi$ is an involution,

(I-2) the product $\varphi\sigma^{-1}$ is an involution,

(FP) neither of the above involutions has fixed points.

Paving: definition

Conditions I-1 and I-2: single face $\longmapsto$ single face, with coherent orientations.

Condition FP: no face or edge is bent onto itself.

The pair of permutations $\langle \alpha, \sigma \rangle$ describes the underlying map H of P.

Why pavings?

Pavings are a natural generalisation of Jones' and Singerman's theory to dimension 3, studied first by Arquès and Koch, Spechner, Lienhardt.

Pavings can be used to efficiently describe solids in computer graphics, including non-convex solids, or bodies with non-trivial topology.

Paving: equivalent definition

A paving P (combinatorially) is a transitive triple $\langle \alpha, \beta, \gamma \rangle$ of involutions without fixed points.

In this case we have that $\sigma = \alpha\beta$ and $\varphi = \gamma\alpha\beta$.

Pavings and free subgroups

In complete analogy with Lemmas 1, we can show that the following statement holds:

Lemma 3

There is a one-to-one correspondence between the set of connected oriented rooted pavings $\mathcal{P}_r(n)$ on n darts and the set of free subgroups of index n in the group $\Delta^+ = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$.

Pavings and conjugacy classes

And the statement below is analogous to Lemma 2:

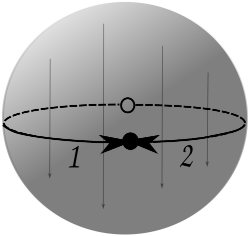
Lemma 4

There is a one-to-one correspondence between the set of isomorphism classes of connected oriented pavings $\mathcal{P}(n)$ on n darts and the set of conjugacy classes of free subgroups of index n in the group $\Delta^+ = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$.

Pavings with ≤ 4 darts

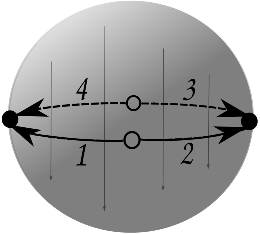
As an example, we can easily classify by hand conjugacy classes of free subgroups of index ≤ 4 in Δ^+ by classifying all pavings with ≤ 4 darts.

Pavings with ≤ 4 darts



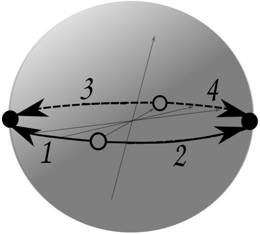
A paving with $(\alpha, \beta, \gamma) \longmapsto ((1,2),(1,2),(1,2))$. Topological $\mathbb{S}^3$.

Pavings with ≤ 4 darts



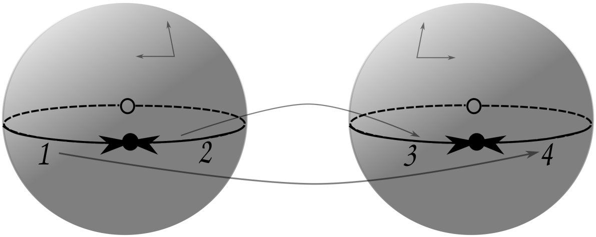
A paving with $(\alpha, \beta, \gamma) \longmapsto ((1,2)(3,4), (1,2)(3,4), (1,3)(2,4))$.
Topological $\mathbb{S}^3$.

Pavings with ≤ 4 darts



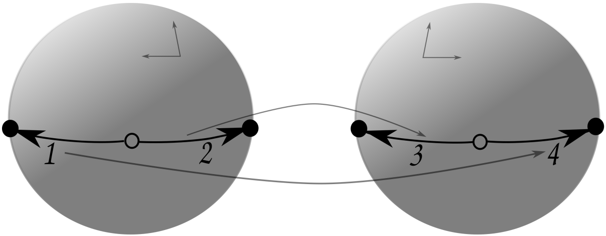
A paving with $(\alpha, \beta, \gamma) \longmapsto ((1,2)(3,4), (1,3)(2,4), (1,2)(3,4))$.
Topological $\mathbb{R}P^3$.

Pavings with ≤ 4 darts



A paving with $(\alpha, \beta, \gamma) \longmapsto ((1,2)(3,4),(1,3)(2,4),(1,3)(2,4))$.
Topological $\mathbb{S}^3$.

Pavings with ≤ 4 darts



A paving with $(\alpha, \beta, \gamma) \longmapsto ((1,2)(3,4), (1,3)(2,4), (1,4)(2,3))$.
Topological $\mathbb{S}^3$.

Topological pavings

In the above examples all but one paving are homeomorphic to the standard three-dimensional sphere $\mathbb{S}^3 = \{(x, y, z, t) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + t^2 = 1\}$. The "special" one is $\mathbb{R}P^3$, that is the quotient of $\mathbb{S}^3$ by central symmetry $(x, y, z) \mapsto (-x, -y, -z)$.

In general: a paving is **not** a manifold (but, obviously, a pseudo-manifold).

Generating series for pavings

The generating series for rooted pavings: $P^\circ(z) = \sum_{n=0}^{\infty} \operatorname{card} \mathcal{P}_r(n)\, z^n = z^2 + 4z^4 + 25z^6 + 208z^8 + 2146z^{10} + 26368z^{12} + 375733z^{14} + 6092032z^{16} + 110769550z^{18} + 2232792064z^{20} + 49426061818z^{22} + 1192151302144z^{24} + \dots$

Its coefficients have asymptotics $\sim 2\sqrt{\frac{2}{\pi}}\, \left(\frac{2}{e}\right)^k\, k^{k+3/2}$, as $n = 2k \to \infty$.

Generating series for pavings

The generating series for the number of isomorphism classes $\mathcal{P}(n)$ of pavings on n darts is $\widetilde{P}(z) = \sum_{n=0}^{\infty} \operatorname{card} \mathcal{P}(n)\, z^n = z^2 + 4z^4 + 11z^6 + 60z^8 + 318z^{10} + 2806z^{12} + 29359z^{14} + 396196z^{16} + 6231794z^{18} + 112137138z^{20} + \dots$

Its coefficients have asymptotics $\sim \sqrt{\frac{2}{\pi}} \left(\frac{2}{e}\right)^k k^{k+1/2}$, as $n = 2k \to \infty$.

Open questions

- The number of e.g. (un-)rooted orientable $(2,3)$-hypermaps with n darts on a fixed genus g surface is $C_1 n^{C_2} \exp(C_3 n)$, with some positive C_i's. Does an analogous for (un-)rooted pavings with n darts hold? (We need to fix the topological type and some measure of complexity for a paving)
- We show that the coefficient sequences of $H^\circ(z)$ and $P^\circ(z)$ are not D–finite. What about those of $\widetilde{H}(z)$ and $\widetilde{P}(z)$? (They may be even not algebro-differential)

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Thank you for your attention!