Sequencing Steiner systems

Daniel Horsley (Monash University)

Padraig Ó Catháin (Dublin City University)

Steiner systems

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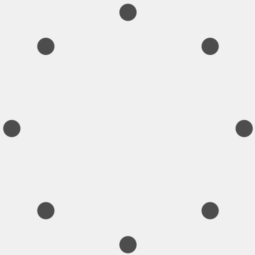

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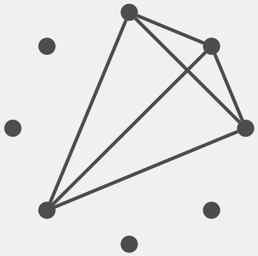

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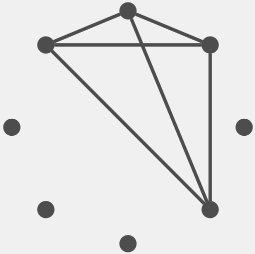

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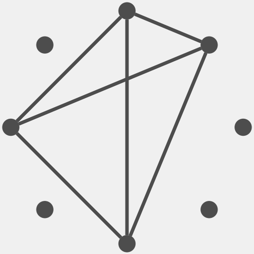

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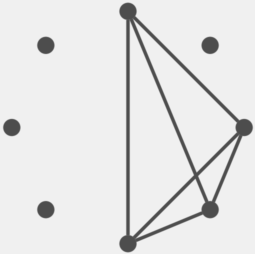

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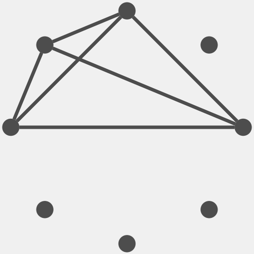

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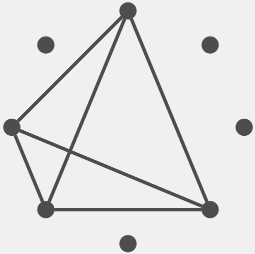

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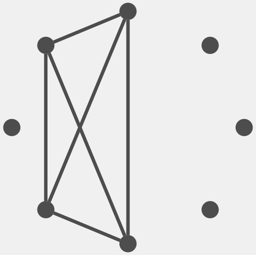

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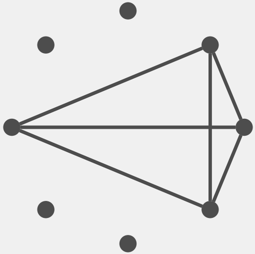

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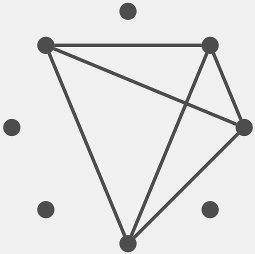

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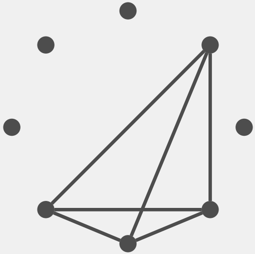

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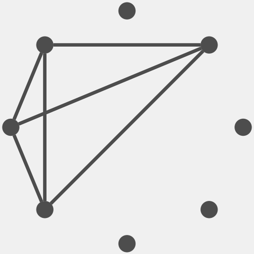

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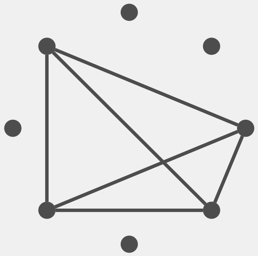

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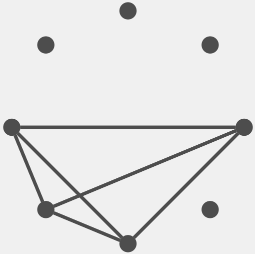

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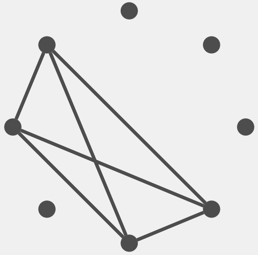

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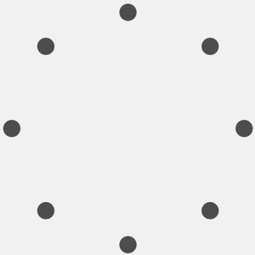

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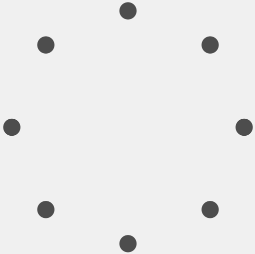

$(n, t+1, t)$-Steiner systems are of particular interest.

$(n, 3, 2)$: *Steiner triple systems* $(n, 4, 3)$: *Steiner quadruple systems*.

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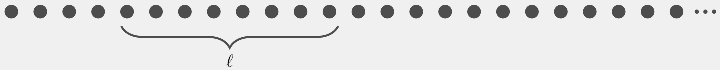

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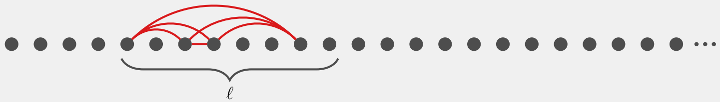

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So precisely if the following does not occur:

It's a *cyclically ℓ-good sequencing* if we include 'wrap-around' sets of consecutive vertices.

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Example for $\ell = 6$:

Assign each set of 6 consecutive vertices weight $\frac{1}{n}$

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Sequencing is harder than colouring, but how much harder?

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Every (n, k, t)-Steiner system has a cyclically ℓ-good sequencing for some $\ell = \Theta(n^{1/2t})$.

(All asymptotic notation in this talk is as $n \to \infty$ with k and t fixed.)

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Kostochka, Mubayi, Verstraëte (2014)

There are $(n, t+1, t)$-Steiner systems with maximum independent set $\Theta((n \log n)^{1/t})$.

Lovász local lemma

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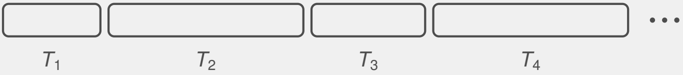

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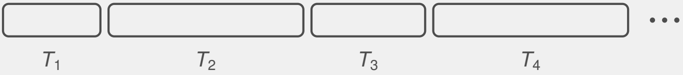

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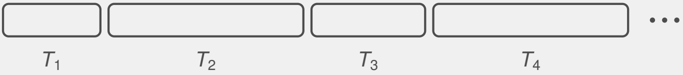

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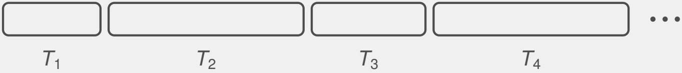

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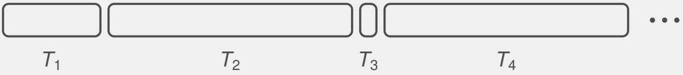

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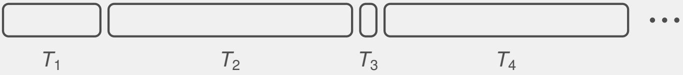

- Possible problems only if some bin has fewer than $\ell - 1$ vertices.

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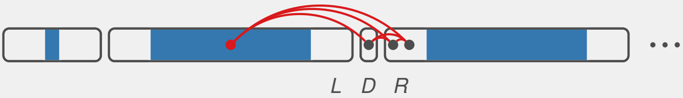

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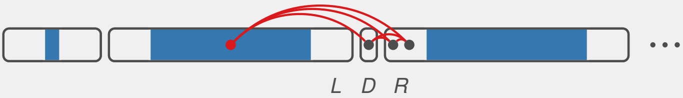

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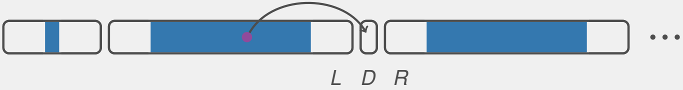

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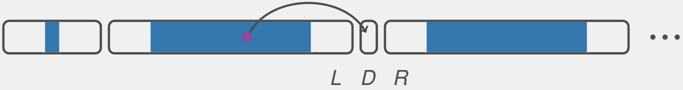

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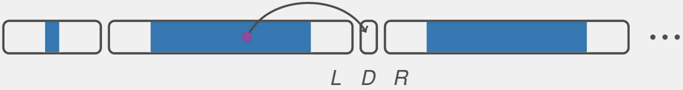

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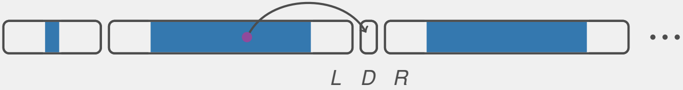

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But for (n, k, t)-Steiner systems with $k \geqslant t + 2$, it's much worse than this.

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Every partial (n, k, t)-Steiner system has a cyclically ℓ-good sequencing for each $\ell \leqslant cn^{1/t}$ where c is the positive real number satisfying

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For $k = 3$, $t = 2$, we have $c \approx 0.0908$.
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  - Our methods apply to general hypergraphs under certain degree conditions.
  - Pavez-Signé/Sanhueza-Matamala/Stein effectively look at the case where ℓ is constant as n grows. (An ℓ-good sequencing of a hypergraph is equivalent to a power of a hamilton cycle in its complement.)

The terminus