Locating Chromatic Number of Infinite Trees

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This talk is based on the following paper :

Yusuf Hafidh, Edy Tri Baskoro, Devi Imulia Dian Primaskun, On the Locating Chromatic Number of Infinite Trees, *under review*

https://doi.org/10.48550/arXiv.2104.04914

Locating Chromatic Number

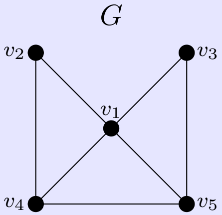
Locating Chromatic number

The locating chromatic number of a simple connected graph G is the smallest integer n, such that G has a **proper** n**-coloring** c and all vertices have **different vector of distances** to the colors generated by c.

Color code

The vector of distances of a vertex v in the locating chromatic number definition is called the color code, denoted by $a_c(v)$.

Example



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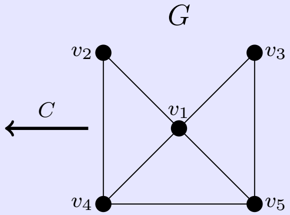

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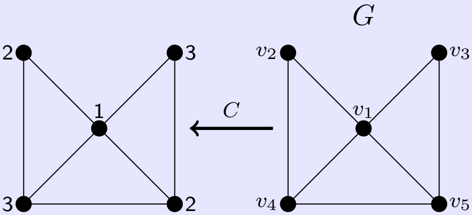

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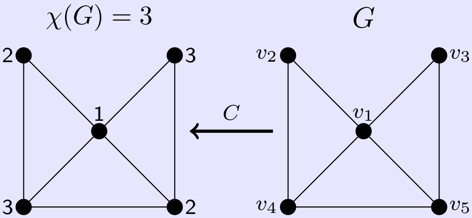

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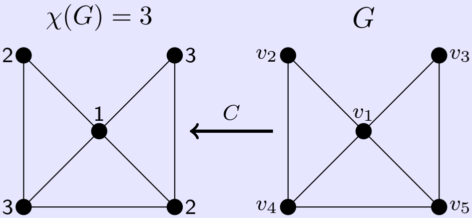

$$a_c(v_1) = (0,1,1)$$
$$a_c(v_2) = (1,0,1)$$
$$a_c(v_3) = (1,1,0)$$
$$a_c(v_4) = (1,1,0)$$
$$a_c(v_5) = (1,0,1)$$

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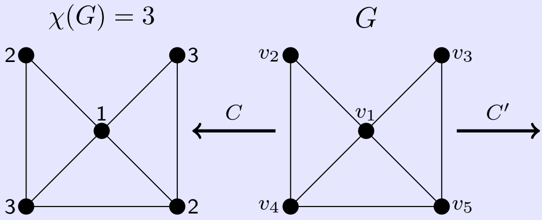

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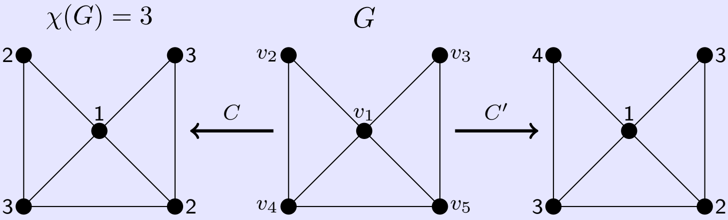

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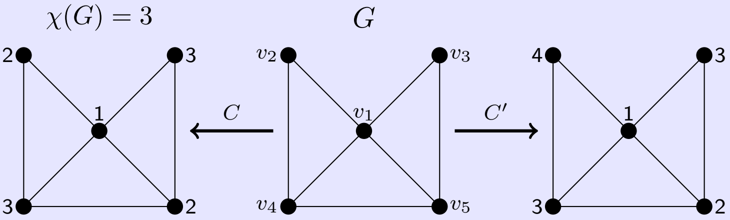

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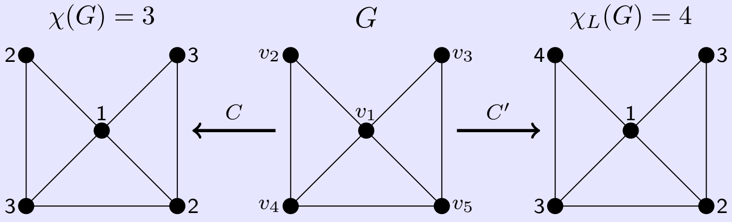

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Connection to metric dimension

Definition (metric dimension)

The metric dimension of a graph G is the size of the smallest ordered set S such that every vertex has unique vector of distances to vertices in S

[1] G. Chartrand, D. Erwin, M.A. Henning, P.J. Slater, P. Zhang, The locating chromatic number of a graph, *Bull. Inst. Combin. Appl.*, 36 (2002), 89–101.

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Theorem [1] : Let G be a graph, then

$$\chi_L(G) \leq dim(G) + \chi(G).$$

χ_L : Locating chromatic number
dim : Metric dimension
χ : Chromatic number

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Infinite graph

Infinite graph

A graph is called an infinite graph if it has infinite vertices.

Finite/infinite dimension

An infinite graph has finite locating chromatic number (metric dimension) if there is an integer k such that $\chi_L(G) = k$ $(dim(G) = k)$. Otherwise we say that it has infinite locating chromatic number (metric dimension).

$$\chi_L(G) \leq dim(G) + \chi(G).$$

Theorem [2]: An infinite tree T has finite metric dimension if and only if T has bounded degree and finite number of branches.

[2] J. Cáceres, C. Hernando, M. Nora, I.M. Pelayo, and M.L. Puertas, On the metric dimensions of infinite graphs, *Electron. Notes Discrete Math.*, **35** (2009) 15–20.

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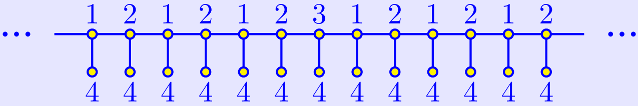
Theorem [2]: An infinite tree T has finite metric dimension if and only if T has bounded degree and finite number of branches.

The characterization is not true for locating chromatic number.



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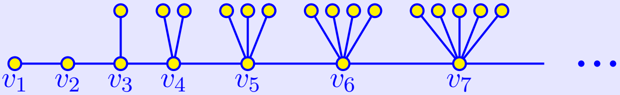
Bounded degree $\neq$ Finite degree



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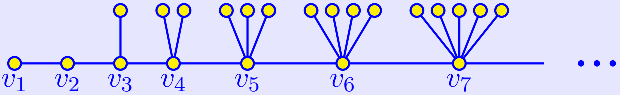


Bounded degree implies finite degree but not otherwise.

Bounded degree is equivalent to Δ exist.

Necessary condition

Theorem

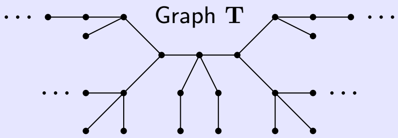
Let G be an infinite graph with finite locating chromatic number, then G has bounded degree.

Theorem [3]: If G is a graph with $\chi_L(G) = k \geq 3$, then $\Delta(G) \leq 4 \cdot 3^{k-3}$.

[3] Y. Hafidh, E.T. Baskoro, On the locating chromatic number of trees, *International journal of mathematics and computer sciences*, 17(2022), 377–394

End-path

In a finite tree T, an **end-path** is a path connecting a leaf to its nearest branch. We generalize the definition of an end-path to also include an infinite path starting from a branch vertex with all vertices other than the branch vertex have degree two in T.



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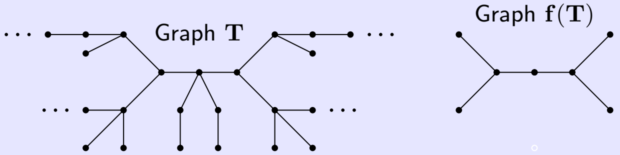

Operation f - Removing end-paths

$f(T)$ is the graph obtained by removing all end-paths of T excluding the branches.

Conjecture

Let T be an infinite tree with bounded degree. Then, $\chi_L(T)$ is finite if and only if there is an integer n such that $f^n(T) = P$ where P is a *path**.

* This *path* includes infinite path, finite path, and K_1.
For $n \geq 2$, $f^n = f^{n-1} \circ f$.

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The $(\Leftarrow)$ part of the conjecture is true.

Theorem

Let T be an infinite tree with bounded degree. If $f^n(T)$ is a *path* for some n, then $\chi_L(T)$ is finite.

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Idea : Prove $\chi_L(f(T))$ is finite implies $\chi_L(T)$ is finite.

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Theorem

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Idea : Prove $\chi_L(f(T))$ is finite implies $\chi_L(T)$ is finite.

1. Start with a locating coloring c of $f(T)$.
2. For every branch b that has an end-path in T, color the i^{th} end-path from b using an alternating $(\chi_L(f(T)) + i, c(b))$ coloring.

We add no more than $\Delta(T)$ colors, so $\chi_L(T) \leq \chi_L(f(T)) + \Delta(T)$.

Necessity

Although the necessity part of the conjecture has not been proven, we will give examples that support the statement.

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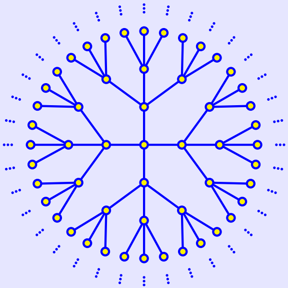
The $(\Rightarrow)$ part of the conjecture : If $\chi_L(T)$ is finite, then there is an integer n such that $f^n(T)$ is a *path*.

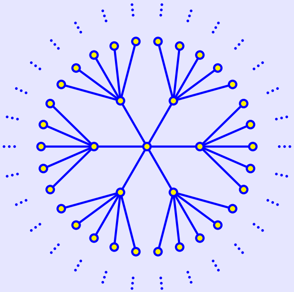
Equivalent : If $f^n(T)$ is not a *path* for all integer n, then $\chi_L(T)$ is infinite.

Two examples of $f^n(T)$ is not a *path* for all integer n:

1. T has no end-path: k-regular tree graph T_k.



Graph T_4

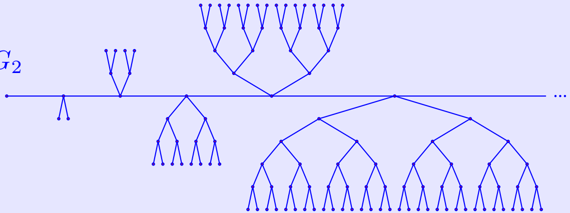


Graph T_6

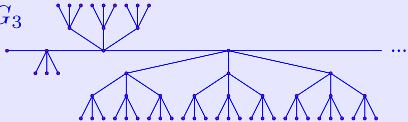
Two examples of $f^n(T)$ is not a *path* for all integer n:

2. $f^k(T) = T$ for some k: Graph G_n below.

G_2



G_3



G_n is the graph obtained from $P_{1\infty}$ by attaching the center of a complete n-ary tree $T(n, i)$ to the i^{th} vertex for all i.

Theorem

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Suppose $\chi_L(T_k) = t < \infty$.
Let v be a vertex and $a_c(v) = (a_1, a_2, \cdots, a_t)$.
Let N be an integer such that for every $n \geq N$, $(2n)^t < k(k-1)^{n-1}$.
Let $M = \max(a_1, a_2, \cdots, a_t, N)$.

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Consider all $k(k-1)^{M-1}$ vertices of distance M to v, $D_M(v)$.
The distance from a vertex in $D_M(v)$ to any color i is at most $2M$.
The number of possible color codes are no more than $(2M)^t$ which is less than the cardinality of $D_M(v)$, a contradiction.

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Theorem

For $n \geq 2$, $\chi_L(G_n) = \infty$.

Thank You