On the DPC partitions of a number

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December 14, 2022

Presentation Order

What is a DPC partition of a number?

A partition of a positive integer n is a nondecreasing sequence of positive integers $(p_1, p_2, \ldots, p_d)$ such that

$$p_1 + p_2 + \ldots + p_d = n.$$

The summands $p_1, p_2, \ldots, p_d$ are called parts of the partition.

It the sequence $(p_{i+1} - p_i)_{i>0} = r, r+1, \ldots \ (r \geq 0)$, then $(p_1, p_2, \ldots, p_d)$ is a *DPC partition* of n.

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Example: The DPC partitions of 16

Example

We are going to calculate $\mathrm{DPC}(16)$.

- $16 = 5 + 5 + 6$. $\quad d = 3,\ k = 0$.
- $16 = 3 + 5 + 8$. $\quad d = 3,\ k = 2$.
- $16 = 1 + 5 + 10$. $\quad d = 3,\ k = 4$.
- $16 = 3 + 3 + 4 + 6$. $\quad d = 4,\ k = 0$.
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The set DPC(n)

We consider the problem of enumerating the set $\mathrm{DPC}(n)$ of partitions of n.

In the previous example:

- $\mathrm{DPC}_E(16) = \{(5,5,6),(3,5,8),(1,5,10),(3,3,4,6)\}$.
- $\mathrm{DPC}_O(16) = \{(4,5,7),(2,5,9)\}$.
- $\mathrm{DPC}(16) = \mathrm{DPC}_E(16) \cup \mathrm{DPC}_O(16)$.
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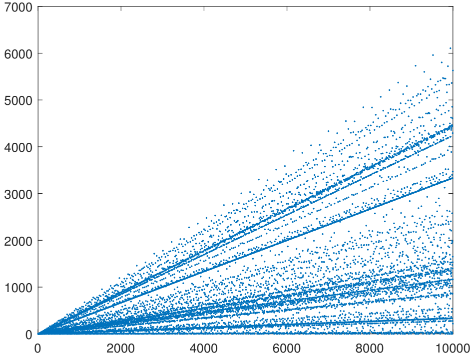
The sequence $(|\mathrm{DPC}(n)|)_{n>0}$

The first terms of the sequence $(|\mathrm{DPC}(n)|)_{n>0}$ are:
$(|\mathrm{DPC}(n)|)_{n>0} = 0, 0, 0, 1, 0, 0, 2, 1, 0, 3, 0, 1, 4, 1, 1, 6, \ldots$



A related partition problem

The set DPC(n) is a generalization of the set AP(n) of partitions of n in which the nondecreasing sequence of parts form an arithmetic progression (AP), that is, the nondecreasing arithmetic progressions of positive integers with sum n.

For instance:

$$\begin{aligned}\mathrm{AP}(10) = \{&(1,\ldots,1),(2,2,2,2,2),(5,5),(10),\\ &(1,9),(2,8),(3,7),(4,6),(1,2,3,4)\}.\end{aligned}$$

Hence $|\mathrm{AP}(10)| = 9$.

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A related partition problem

We will follow the methods in the papers below to study the DPC partitions problem.

- F. J. de Vega, *An extension of Furstenberg's theorem of the infinitude of primes*, JP J. Algebra Number Theory Appl. 53(1) (2022), 21-43.
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Presentation Order

The main idea

The usual divisors trivially solve the problem of the partition of a number into equal parts.

Now, for each $k \in \mathbb{Z}$, we will consider a new product mapping $(\odot_k)$ that will generate an arithmetic similar to the usual one.

In this new arithmetic, the divisors of an integer n will trivially solve the problem of the representation of n as the sum of an integer sequence whose differences between consecutive parts are consecutive integers.

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Step 1. DPC_k product $\odot_k$

For each $k \in \mathbb{Z}$, we will consider a new product mapping ($\odot_k$) that will generate an arithmetic (DPC_k-*arithmetic*) similar to the usual one:

Definition

Given $m, k \in \mathbb{Z}$, for all positive integers n, we define the following expression

$$\begin{aligned} m \odot_k n = &(m-n+1) + (m-n+1+k) + (m-n+1+k+k+1) \\ &+ \ldots + (m-n+1+k+k+1+\ldots+k+n-2) \end{aligned}$$

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Step 1. DPC_k product $\odot_k$

We can extend the definition for all integer n adding the previous expression:

$$m \odot_k n = (m-n+1)\cdot n + \frac{(n-1)\cdot n \cdot k}{2} + \frac{(n-2)\cdot(n-1)\cdot n}{6}.$$

In connection with the above result, for each $k \in \mathbb{Z}$, the expression "*given a* DPC_k*-arithmetic*" refers to the fact that we are going to work with integers, the sum, the new product and the usual order.

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Step 2a. DPC_k-*arithmetic* divisor

Definition

Given a DPC_k-*arithmetic*, an integer $d > 0$ is called a divisor of a $(arith\ \mathrm{DPC}_k)$ if there exists some integer b such that $a = b \odot_k d$. We can write:

$$d \mid a\ (arith\ \mathrm{DPC}_k) \Leftrightarrow \exists b \in \mathbb{Z} \text{ such that } b \odot_k d = a.$$

In other words, d is the number of terms of the summation that represents the DPC_k-*arithmetic* product. For instance: $6 \odot_1 4 = 3 + 4 + 6 + 9 = 22$.

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Step 2b. DPC_k-*arithmetic* quotient $\oslash_k$

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Given a DPC_k-*arithmetic*, an integer c is called a quotient of a divided by b ($arith\ \mathrm{DPC}_k$) if and only if $c \odot_k b = a$. We write: $a \oslash_k b = c \Leftrightarrow c \odot_k b = a$.

Also, we can express the DPC_k-*arithmetic* quotient with the usual one:

$$a \oslash_k b = \frac{a}{b} + (b-1) \cdot (1 - \frac{k}{2} - \frac{b-2}{6}).$$

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The connection between the previous points are represented by the following corollary:

Corollary

Let $d > 0$, d is a divisor of a (arith DPC_k), denoted by $d \mid_k a$, if and only if $a \oslash_k d$ is an integer.

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Step 3. The set of divisors

To characterize the set of divisors of n in a DPC_k-*arithmetic*, denoted by $\mathrm{D}_k(n)$, we have the following lemmas:

Notation

We write the set of even and odd numbers as follows:

- $E = \{\ldots, -4, -2, 0, 2, 4, 6, \ldots\}$.
- $O = \{\ldots, -3, -1, 1, 3, 5, 7, \ldots\}$.

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Lemma (1)

Given a DPC_k-arithmetic and $k \in O$, the divisors of $a \in \mathbb{Z}$ (arith DPC_k), denoted by $D_O(a)$, satisfy the following conditions:

1. *If $d \mid a$ then $d \mid_k a \Leftrightarrow 2 \nmid d \wedge 3 \nmid d$.*
2. *If $d \mid 2a \wedge d \nmid a$ then $d \mid_k a \Leftrightarrow 2 \mid d \wedge 3 \nmid d$.*
3. *If $d \mid 3a \wedge d \nmid a$ then $d \mid_k a \Leftrightarrow 2 \nmid d \wedge 3 \mid d \wedge \frac{3a}{d} \equiv 1 \pmod 3$.*
4. *If $d \mid 6a \wedge d \nmid 2a \wedge d \nmid 3a$ then $d \mid_k a \Leftrightarrow 2 \mid d \wedge 3 \mid d \wedge \frac{6a}{d} \equiv 5 \pmod 6$.*

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4. *If $d \mid 6a \wedge d \nmid 2a \wedge d \nmid 3a$ then $d \mid_k a \Leftrightarrow 2 \mid d \wedge 3 \mid d \wedge \frac{6a}{d} \equiv 5 \pmod 6$.*

Example

Express the number 20 in all possible ways as a sum of a sequence $(a_n)_{n>0}$ verifying $a_{i+1} - a_i = i + 2$, $i \in \mathbb{N}$.

Solution. As the first difference is 3, we need to calculate the the set $\mathrm{D}_3(20)$.

- $\mathrm{Div}(20) = \{1, 2, 4, 5, 10, 20\}$. By Point 1 of Lemma 1, only 1 and 5 are divisors of 20 (*arith* DPC_3).
- $\mathrm{Div}(40) \setminus \mathrm{Div}(20) = \{8, 40\}$. By Point 2 of the previous lemma, 8 and 40 are divisors of 20 (*arith* DPC_3).
- $\mathrm{Div}(60) \setminus \mathrm{Div}(20) = \{3, 6, 12, 15, 30, 60\}$. By Point 3, the candidates are 3 and 15 but $60/3 = 20 \equiv 2 \pmod 3$ and $60/15 = 4 \equiv 1 \pmod 3$. Hence $3 \nmid_3 20$ and $15 \mid_3 20$.
- $\mathrm{Div}(120) \setminus (\mathrm{Div}(60) \cup \mathrm{Div}(40)) = \{24, 120\}$. By Point 4, $24 \mid_3 20$ because $120/24 = 5 \equiv 5 \pmod 6$ but $120 \nmid_3 20$ because $120/120 = 1 \equiv 1 \pmod 6$.

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Hence $\mathrm{D}_3(20) = \{1, 5, 8, 15, 24, 40\}$ and the solution is:

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Step 3. The set of divisors

Lemma (2)

Given a DPC_k-arithmetic and $k \in E$, the divisors of $a \in \mathbb{Z}$ (arith DPC_k), denoted by $D_E(a)$, satisfy the following conditions:

1. *If $d \mid a$ then $d \mid_k a \Leftrightarrow 3 \nmid d$.*
2. *If $d \mid 3a \wedge d \nmid a$ then $d \mid_k a \Leftrightarrow 3 \mid d \wedge \frac{3a}{d} \equiv 1 \pmod 3$.*

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Theorem

Given a DPC_k-arithmetic, the set of primes is:

1. $\{2^{2s-1} \cdot 3^{t-1} : s, t \in \mathbb{N}\}$ *if* $k \in O$.
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Example

Proof that the number 1944 can be expressed uniquely as a sum of a sequence $(a_n)_{n>0}$ whose differences between their parts satisfy $a_{i+1} - a_i = i$, $i \in \mathbb{N}$.

Solution. $1944 = 2^3 \cdot 3^5$. By the previous theorem, the divisors (*arith* DPC_1) are 1 and 2^4. Hence, $1944 \oslash_1 16 = 94$ and the solution is:
$1944 = 94 \odot_1 16 = 79 + 80 + 82 + \ldots + 170 + 184 + 199.$

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Step 5. Divisors $\Leftrightarrow$ Integer Sequences

If we have the divisors of n ($arith$ DPC_k), then we have the integer sequences (with sum n) whose differences between consecutive terms are the consecutive integers $k, k+1, \ldots$.

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Example

Express the number 12 in all possible ways as a sum of a sequence $(a_n)_{n>0}$ verifying $a_{i+1} - a_i = i + 1$, $i \in \mathbb{N}$.

Solution. As the first difference is 2, we need to calculate the the set $\mathrm{D}_2(12)$.

- $\mathrm{Div}(12) = \{1, 2, 3, 4, 6, 12\}$. By Point 1 of Lemma 2, 1, 2 and 4 are divisors of 12 (*arith* DPC_2).
- $\mathrm{Div}(36) \setminus \mathrm{Div}(12) = \{9, 18, 36\}$. By Point 2 of the previous lemma, 9 and 36 are divisors of 12 (*arith* DPC_2) because $36/9 = 4 \equiv 1 \pmod 3$ and $36/36 = 1 \equiv 1 \pmod 3$. But $18 \nmid_2 12$ because $36/18 = 2 \equiv 2 \pmod 3$.

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Hence, $\mathrm{D}_2(12) = \{1, 2, 4, 9, 36\}$ and the nontrivial solutions are:

- $d = 2 \Rightarrow 12 \oslash_2 2 = 6 \Rightarrow 12 = 6 \odot_2 2 \Rightarrow 12 = 5 + 7.$
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- $d = 9 \Rightarrow 12 \oslash_2 9 = -8 \Rightarrow 12 = -8 \odot_2 9 \Rightarrow 12 = -16 - 14 - 11 - \ldots + 11 + 19 + 28.$
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Step 6. First term > 0

We can assess when the first term of the representations of the previous examples is positive and therefore solve the proposed partition problem:

If $d \in \mathrm{D}_k(n)$, we can calculate the first term of the partition doing $n \oslash_k d - d + 1$. If the first term is greater than 0 we have the following expression:

$$\frac{n}{d} + (d-1)(1 - \frac{k}{2} - \frac{d-2}{6}) - d + 1 > 0 \Leftrightarrow k < \frac{2n}{d(d-1)} - \frac{d-2}{3}$$

Notation:

$$k_d = \frac{2n}{d(d-1)} - \frac{d-2}{3}$$

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If $d \in \mathrm{D}_k(n)$, we can calculate the first term of the partition doing $n \oslash_k d - d + 1$. If the first term is greater than 0 we have the following expression:

$$\frac{n}{d} + (d-1)(1 - \frac{k}{2} - \frac{d-2}{6}) - d + 1 > 0 \Leftrightarrow k < \frac{2n}{d(d-1)} - \frac{d-2}{3}$$

Notation:

$$k_d = \frac{2n}{d(d-1)} - \frac{d-2}{3}$$

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Step 7. Length of the partitions

The previous expression allows us to make some remarks about the length of the partitions of $\mathrm{DPC}(n)$:

Remark (1)

If $k \in \mathcal{O}$, $k \geq 0$, we have only to study the divisors $d \in D_{\mathcal{O}}(n)$ such that $2 < d \leq \lfloor \sqrt[3]{6n} \rfloor$.

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Based on the above, we can count the partitions distinguishing the cases $k \in \mathrm{E}$ and $k \in \mathrm{O}$:

Example

Calculate $|\mathrm{DPC}(100)|$.

- $k \geq 0$, $k \in \mathrm{O}$:
  $\mathrm{D_O}(100) = \{1, 3, 5, 8, 25, 40, 75, 120, 200\}$. Based on Remark 1, we have to study the divisors d such that $2 <$d$\leq \lfloor \sqrt[3]{600} \rfloor = 8$.

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d	k_d	$k \in \mathrm{O}$ and $0 \leq k < k_d$	# DPC partitions
3	33	$k = 1,\ k = 3,\ \ldots,\ k = 29,\ k = 31$	16
5	9	$k = 1,\ k = 3,\ k = 5,\ k = 7$	4
8	$1.57\ldots$	$k = 1$	1

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- $k \geq 0$, $k \in \mathrm{E}$: $\mathrm{D_E}(100) = \{1, 2, 3, 4, 5, 10, 12, 20, 25, 30, 50, 75, 100, 300\}$.
  Based on Remark 2, we have to study the divisors d such that $2 < d \leq \lfloor \sqrt[3]{600} \rfloor + 1 = 9$.

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4	16	$k = 0,\ k = 2,\ \dots,\ k = 12,\ k = 14$	8
5	9	$k = 0,\ k = 2,\ k = 4,\ k = 6,\ k = 8$	5

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Hence $|\mathrm{DPC}(100)| = 17 + 8 + 5 + 16 + 4 + 1 = 51$.

If we want to calculate a concrete partition, for instance $d = 5$, $k = 8$, we can do:

$100 \oslash_8 5 = 100/5 + 4(1 - 8/2 - 3/6) = 6 \Rightarrow 100 = 6 \odot_8 5 = 2 + 10 + 19 + 29 + 40$.

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Step 9. The floor and ceiling functions

We can use the floor and the ceiling functions to count the even and odd numbers in each case.

Remark (3)

The number of nonnegative even numbers less than a real $x > 0$ is given by the expression $\lfloor \frac{1}{2} \lceil x+1 \rceil \rfloor$.

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Step 10. The main theorem

With all of the above, we can prove the following theorem:

Theorem

Given a positive integer n, then the cardinality of the set $DPC(n)$, denoted by $|DPC(n)|$, is equal to

$$\sum_{\substack{d \,\in\, D_E(n) \\ 2 < d \leq \lfloor \sqrt[3]{6n} \rfloor + 1}} \left\lfloor \frac{1}{2} \left\lceil \frac{2n}{d(d-1)} - \frac{d-5}{3} \right\rceil \right\rfloor + \sum_{\substack{d \,\in\, D_O(n) \\ 2 < d \leq \lfloor \sqrt[3]{6n} \rfloor}} \left\lfloor \frac{1}{2} \left\lceil \frac{2n}{d(d-1)} - \frac{d-2}{3} \right\rceil \right\rfloor .$$

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Presentation Order

Corollary

$|DPC(3^n)| = 0, n \in \mathbb{N}.$

Proof.

The divisors of 3^n (*arith* DPC_k) are: $\mathrm{D_E}(3^n) = \{1, 3^{n+1}\}$ and $\mathrm{D_O}(3^n) = \{1, 2, 3^{n+1}\}$. Based on Remarks 1 and 2, there are no partitions. $\square$

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Corollary

Let $a = 3^n \cdot p$ *where* $p \equiv 5 \pmod 6$ *is prime. If* $p > \sqrt[3]{6a}$ *and* $2 \cdot 3^{n+1} > \sqrt[3]{6a}$*, then* $|DPC(a)| = 0$.

Proof.

Based on Lemmas 1 and 2, $\mathrm{D_E}(a) = \{1, p, 3a\}$ and $\mathrm{D_O}(a) = \{1, 2, p, 2 \cdot 3^{n+1}, 2p, 3a\}$. The smallest divisors could be p or $2 \cdot 3^{n+1}$. By Remark 1, the result follows. □

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As $\mathrm{D_E}(p) = \{1, p, 3p\}$ and $\mathrm{D_O}(p) = \{1, 2, 6, p, 2p, 3p\}$, If $p > 29$ then $d = 6$ produces at least one partition because $6 < \sqrt[3]{6p}$. For instance, if $p = 41$ then $\sqrt[3]{6 \cdot 41} \simeq 6.26$. Hence $d = 6$ produces a partition of $\mathrm{DPC}(41)$ $(1 + 2 + 4 + 7 + 11 + 16 = 41)$ but $d = 6$ does not produce a partition of 29 because $\sqrt[3]{6 \cdot 29} \simeq 5.58$. □

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Presentation Order

Sums of consecutive triangular numbers

We consider the problem of the representation of a positive integer a as a sum of d consecutive triangular numbers. In particular, we focus on the case that a is a triangular number too.

$$\triangle_l + \triangle_{l+1} + \ldots + \triangle_{l+d-1} = a. \tag{1}$$

As $\triangle_{l+1} - \triangle_l = l + 1$, we consider the DPC_{l+1}-*arithmetic*, $l \in \mathbb{N}$. If a is the sum of d consecutive triangular numbers then $d \mid_{l+1} a$. The first addend is $a \oslash_{l+1} d - d + 1$. Hence, $a \oslash_{l+1} d - d + 1 = \triangle_l$ implies

$$l^2 + dl + \frac{(d-1)(d+1)}{3} - \frac{2a}{d} = 0. \tag{2}$$

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Sums of consecutive triangular numbers

The discriminant of (2) must be a perfect square, hence for some integer h

$$\frac{(2-d)(2+d)}{3} + \frac{8a}{d} = h^2. \tag{3}$$

If a is a triangular number too then $8a+1$ is a perfect square. By (3), for some integer s, we have the following equation:

$$dh^2 + \frac{(d-1)(d^2+d-3)}{3} = s^2. \tag{4}$$

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Sums of consecutive triangular numbers

Equation (4) appears in

- M. McMullen, *Playing with blocks*, Math Horiz. 25(4) (2018), 14–15.
- D. Subramaniam, E. Treviño and P. Pollack, *On sums of consecutive triangular numbers*, In: Proceedings of the Integers Conference 2018, Integers 20A (2020), #A15.

but we can combine (3) and (4) to obtain extra elementary results. For instance:

Sums of consecutive triangular numbers

- If $d = 2$, from (4) we obtain the original Pell equation $s^2 - 2h^2 = 1$. Doing $d = 2$ in (3) we obtain $4a = h^2$. This equation leads us to the solution of Pell equation right away: a must be a triangular square number. See sequence A001110 in the OEIS.

  The fact that each triangular square number produces a solution on Pell equation is well known. For instance, if $a = 36$, then $h = 12$ and $s = 17$. Hence $l = 5$ and $36 = \triangle_5 + \triangle_6$.

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Sums of consecutive triangular numbers

- If $d = 3$, from (4) we obtain the equation $s^2 - 3h^2 = 6$. Similar to the previous case, based on (3), $(8a - 5)/3 = h^2$ hence $a \in \{1, 4, 10, 19, 31, 46, \ldots\}$ (see A005448).
  The triangular numbers in this sequence can be written as a sum of 3 consecutive triangular numbers:
  $10, 136, 1891, 26335, \ldots$ (see A128862).

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Sums of consecutive triangular numbers

- If $d = 4$, $s^2 - 4h^2 = 17$. Since 17 is prime, we can factor the left side of the equation to get $h = \pm 4$, $s = \pm 9$, $a = 10$ and $l = 0$ or $l = -4$. This means that if we admit negative triangular numbers $(\triangle_{-n} = -n \cdot (-n+1)/2)$, 10 is the unique triangular that can be written as a sum of 4 consecutive triangular numbers in two different ways: $10 = \triangle_0 + \triangle_1 + \triangle_2 + \triangle_3 = \triangle_{-4} + \triangle_{-3} + \triangle_{-2} + \triangle_{-1}$. In our original problem $l > 0$, hence we have no solution when $d = 4$. To end this case, if we use (3) then $2a - 4 = h^2$. Hence the unique triangular number a such that $2a - 4$ is a perfect square is $a = 10$.

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Sums of consecutive triangular numbers

For the rest of the cases, we can solve the Pell-like equation from (4) with the help of (3). If $d > 4$ is a square, we can solve (4) by factoring. Subramaniam et al. saw that for almost all values of d there are no solutions.

With the approach of our paper, we can solve the main problem differently: given a triangular number a, we can know whether or not it is the sum of consecutive triangular numbers by calculating the divisors of a using Lemmas 1, 2 and Remarks 1, 2.

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Example

Express the triangular number 799480 in all different ways as a sum of consecutive triangular numbers.

Solution. The divisors $d \in \mathrm{D_E}(799480)$, $d \le \lfloor \sqrt[3]{6 \cdot 799480} \rfloor + 1 = 169$, are: $1, 3, 5, 11, 16, 23, 48, 55, 79, 80, 115, 165$. The unique not trivial divisor that produces an square in (3) is $d = 5$. Hence, $h = 1131$ and $l = 563$. Then,

$$799480 = \triangle_{563} + \triangle_{564} + \triangle_{565} + \triangle_{566} + \triangle_{567}. \tag{5}$$

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$110, 115, 120, 138, 158, 165.$

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Presentation Order

Conclusions

- The novel way of studying a partition problem by calculating the divisors of a number in an arithmetic similar to the usual one is the main contribution of this talk.
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Conclusions

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Thank you very much!