There are infinitely many Hilbert cubes of dimension 3 in the set of squares

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The Hilbert cube of dimension d is the set of integers

$$H(a_0; a_1, \ldots, a_d) = a_0 + \{0, a_1\} + \ldots + \{0, a_d\} = \left\{ a_0 + \sum_{i=1}^{d} \varepsilon_i a_i : \varepsilon_i \in \{0, 1\} \right\}.$$

Brown, Erdős and Freedman asked whether the maximal dimension of a Hilbert cube in the set $S = \{n^2 : n \in \mathbf{N}\}$ of integer squares is absolutely bounded or not. Dietmann and Elsholtz proved that if $H(a_0; a_1, \ldots, a_d) \subset S \cap [0, N]$, then $d \leq 7 \log\log N$ for all sufficiently large values of N. In the opposite direction, we prove that there are infinitely many Hilbert cubes of dimension 3 in the set of squares. Our method allow us to prove that the number of integers $a_0, a_1, a_2, a_3 \in [0, N]$ such that the Hilbert cube $H(a_0; a_1, a_2, a_3)$ is contained in the set of squares is $\gg N^{1/8}$. Moreover, we prove that for each $i, j \in \{0, 1, 2, 3\}$ with $i < j$, the set

$$\left\{ \frac{a_i}{a_j} : H(a_0; a_1, a_2, a_3) \subset S \right\}$$

is dense in the set of positive real numbers (in the Euclidean topology).